BHARATHIDHASANAR MATRIC HIGHER SECONDARY SCHOOL

ARAKKONAM

XII – MATHEMATICS

MATERIAL

6 Marks & 10 Marks

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APPLICATIONS OF MATRICES AND DETERMINANTS

Find the adjoint of matrices:

(i)$\begin{bmatrix} 3 & -1 \\ 2 & -4 \end{bmatrix}$; (ii) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{pmatrix}$;$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$.

Solution: (i) A=$\begin{bmatrix} 3 & -1 \\ 2 & -4 \end{bmatrix}$.

the matric of cofactor$[Aij]$=$\begin{bmatrix} -4 & -2 \\ 1 & 3 \end{bmatrix}$

Therefore adjA=$(Aij)^T$=$\begin{bmatrix} -4 & 1 \\ -2 & 3 \end{bmatrix}$

(ii) A=$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 5 \end{bmatrix}$

Cofactor of 1 is =+ (15-3) =15

Cofactor of 2 is =- (0-0) = 0

Co-factor of 3 is =+ (0-10) =-10

cofactor of 0 is =-(6-12) = 6

cofactor of 5 is =+(3-6) -3

cofactor of 0 is =-(4-4) =0

cofactor of 2 is =+(0-15) =-15

cofactor of 4 is =-(0-0) =0

cofactor of 3 is =+(5-0) = 5

$$Aij = \begin{bmatrix} 15 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}$$

There fore adj.A $= \begin{bmatrix} 15 & 6 & -15 \\ 0 & 3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$

(iiI) A= $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

cofactor of 2 is = + (1-4) = -3

cofactor of 5 is = - (3-2) = -1

cofactor of 3 is =+ (6-1) = 5

cofactor of 3 is =- (5-6) = 1

cofactor of 1 is =+ (2-3) = -1

cofactor of 2 is =- (4-5) = 1

cofactor of 1 is =+ (10-3) = 7

cofactor of 2 is =- (4-6) = 5

cofactor of 1 is =+ (2-15) = -13

$$Aij = \begin{bmatrix} -3 & -1 & 5 \\ 1 & -1 & 1 \\ 7 & 5 & -13 \end{bmatrix}$$

There fore adj.A $=\begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$

2.Find the adjoint of the matrix A=$\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$. and

verify the result. A(adj.A) = (adj.A)A = $|A|I_2$

Solution A = $\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$.

the matrix of cofactor$[Aij]$=$\begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}$

There fore adjA=$(Aij)^T$=$\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$

A(adj.A) $=\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix} = -11\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A|I_2$

(adj.A)A $=\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$

$= \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix} = -11\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A|I_2$

Hence A(adj.A) = (adj.A)A = $|A|I_2$

3.find the adjoint of matrix A = $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ and verify the result.

A(adj.A) = (adj.A)A = |A|I.

Solution: $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix}$

=3(-3+4) + 3(2-0) +(-2-0)

=3+6-8=1

Cofactor of 2 is =+ (-3+4) = 1

Cofactor of 5 is =- (2-0) = -2

cofactor of 3 is =+ (-2-0) = -2

cofactor of 3 is =- (-3+4) = -1

cofactor of 1 is =+ (3-0) = 3

cofactor of 2 is =- (-3+0) = 3

cofactor of 1 is =+ (-12+12) = 0

cofactor of 2 is =- (12-8) =-4

cofactor of 1 is =+ (-9+6) = -3

$Aij = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}$

Therefore adj.A = $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

A(adj.A) = $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ = (1)I = A I

(adj.A)A = $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ = (1)I = $|A|$ I

Hence A(adj.A) = (adj.A)A = $|A|I_3$Hence proved.

4.Find the inverse of each of the following matrices:

(i) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$,(ii) $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$,$(iii)$ $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$,

(iv) $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$,$(v)$ $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Solution: (i)A $= \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$

=1(1-1)-0(2+1) +3(-2-1)

=-9 ≠0

Co-factor of 1 is =+ (1-1) = 0

Co-factor of 0 is =- (2+1) = -3

Co-factor of 3 is =+ (-2-1) = -3

Co-factor of 2 is =- (0+3) = -3

Co-factor of 1 is =+ (1-3) = -2

Co-factor of -1 is =- (-1-0) = 1

Co-factor of 1 is =+ (0-3) = -3

Co-factor of -1 is =- (-1-6) = 7

Co-factor of 1 is = + (1-0) = 1

$$Aij = \begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 1` \\ -3 & 7 & 1 \end{bmatrix}$$

There fore adj.A $=(Aij)T = \begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix}$

$A^1 = \frac{1}{|A|}$(adj.A) $= \frac{1}{-9}\begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$

(ii) Solution: (i)A $= \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix}$

=1(2-6) -3 (4-3) +7(8-2)

=-4-3+42=35≠0

Cofactor of 1 is =+ (2-6) = -4

co -factor of 3 is =- (4-3) = -1

co- factor of 7 is =+ (8-2) = 6

co-factor of 4 is =- (3-14) = 11

co-factor of 2 is =+ (1-7) = -6

co-factor of 3 is =- (2-3) = 1

co-factor of 1 is =+ (9-14) = -5

co-factor of 2 is =- (3-28) = 25

co-factor of 1 is =+ (2-12) = -10

$$Aij = \begin{bmatrix} -4 & -1 & 6 \\ 11 & -6 & 1 \\ -5 & 25 & -10 \end{bmatrix}$$

$$\text{Therefore adj.A } = \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{A}(\text{adj.A}) = \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

(iii) Solution: (iii)A $= \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

=1(3-0) -2(-1-0) +(2-0)

=3+2-4=1

Cofactor of 1 is =+ (3-0) = 3

Cofactor of 2 is =- (-1-0) = 1

cofactor of -2 is =+ (2-0) = 2

cofactor of -1 is =- (4-0) = -4

cofactor of 3 is =+ (1-0) = 1

cofactor of 0 is =- (-2-0) = 2

cofactor of 0 is =+ (0+6) = 6

cofactor of -2 is =- (0-2) = 2

cofactor of 1 is = + (3+2) = 5

Aij = $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}$

(adj. A) =(Aij)T= $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

A^{-1} = $\frac{Adj\ .A}{|A|}$ = $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

(iv) Solution: (iv) A = $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$

$$|A| = \begin{vmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{vmatrix}$$

=8(-4+2) + 1(20-20) -3(5-10)

=-16+0+15=-1

Cofactor of 8 is =+ (-4+2) = -2

Cofactor of -1 is =- (20-20) = 0

cofactor of -3 is =+ (3-10) = -5

cofactor of -5 is =- (4-3) = -1

cofactor of 1 is =+ (-32+30) = -2

cofactor of 2 is =- (-8+10) = -2

cofactor of 10 is =+ (-2+3) = 1

cofactor of -1 is =- (16-15) = -1

cofactor of -4 is =+ (8-5) = 3

$$[A_{ij}] = \begin{bmatrix} -2 & 0 & -5 \\ -1 & -2 & -2 \\ 1 & -1 & 3 \end{bmatrix}$$

$$(\text{adj. } A) = (Aij)T = \begin{bmatrix} -2 & -1 & 1 \\ 0 & -2 & -1 \\ -5 & -2 & 3 \end{bmatrix}$$

$A^{-1} = \frac{1}{|A|}(\text{adj . A}) = \frac{1}{-1}\begin{bmatrix} -2 & -1 & 1 \\ 0 & -2 & -1 \\ -5 & -2 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}(\text{adj . A}) = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$

(v) Solution: (v) $|A| = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix}$

=2(6-2) -2(2-1) +1(2-3)

=8-2-1=5

Cofactor of 2 is =+ (6-2) = 4

Cofactor of 2 is =- (2-1) = -1

cofactor of 1 is =+ (2-3) = -1

cofactor of 1 is =- (2-2) = -2

cofactor of 3 is =+ (4-1) = 3

cofactor of 1 is =- (4-2) = -2

cofactor of 1 is =+ (2-3) = -1

cofactor of 2 is =- (2-1) = -2

cofactor of 2 is =+ (6-2) = 4

$$[A_{ij}] = \begin{bmatrix} 4 & -1 & -1 \\ -2 & 3 & -2 \\ -1 & -1 & 4 \end{bmatrix}$$

$$(\text{adj. } A) = (Aij)T = \begin{bmatrix} 4 & -2 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj . } A) = \frac{1}{5}\begin{bmatrix} 4 & -2 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 4 \end{bmatrix}$$

5. if $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ verify that (i) $(AB)^{-1} = B^{-1}A^{-1}$

(ii) $(AB)^T = B^T A^T$

Solution: (i) $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 10-2 & -5-2 \\ 14-3 & -7+3 \end{bmatrix}$$

$= \begin{bmatrix} 8 & -3 \\ 11 & -4 \end{bmatrix}$

To find A 1

$$|A| = \begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix} = 15\text{-}14 = 1$$

$[Aij] = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

$$\text{adj } .A = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} (\text{adJ } .A) = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

To find B^{-1}

$$|B| = \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 2\text{-}1 = 1$$

$[Bij] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$$\text{adj } . B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} (\text{adj } .B) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$B^1 \; A^1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$

$= \begin{bmatrix} 3-7 & -2+5 \\ 3-14 & -2+10 \end{bmatrix}$ …………………… 1

To find $(AB)^{-1}$

$|AB| = \begin{vmatrix} 8 & -3 \\ 11 & -4 \end{vmatrix}$ =32+33 =1

Matrix of cofactor of(AB) = $\begin{bmatrix} -4 & -11 \\ 3 & 8 \end{bmatrix}$

Therefore adj.(AB) = $\begin{bmatrix} -4 & 3 \\ -11 & 8 \end{bmatrix}$

Therefore $(AB)^{-1} = \frac{1}{|A|}$ (adj AB)= $\begin{bmatrix} -4 & 3 \\ -11 & 8 \end{bmatrix}$

From (1) and) (2) $(AB)^T = B^T A^T$

(ii) $(AB)^T = \begin{bmatrix} 8 & 11 \\ -3 & -4 \end{bmatrix}$ ……………………………………… (3)

Also $B^T A^T = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 10-2 & 14-3 \\ -5+2 & -7+3 \end{bmatrix} = \begin{bmatrix} 8 & 11 \\ -3 & -4 \end{bmatrix}$ ……….(4)

From (3) and (4) we get $(AB)^T = B^T A^T$

6.find the inverse of the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix}$

=3(-3+4) +3(2-0) +4(-2-0)

=3+6-8=1

Cofactor of 3 is =+ (-3+4) = 1

Cofactor of -3 is =- (2-0) = -2

Cofactor of 4 is =+ (-2-0) = -2

Cofactor of 2 is =- (-3+4)= -1

cofactor of -3 is =+ (3-0)= 3

cofactor of 4 is =- (-3-0) = 3

cofactor of 0 is =+ (-12+12) = 0

cofactor of -1 is =- (12-8) = -4

cofactor of 1 is =+ (-9+6) = -3

$$Aij = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}$$

$$(adj.\ A) = (Aij)T = \begin{bmatrix} -1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(adj.\ A) = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

To find that $A^3 = A^{-1}$

$$A^2 = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9-6+0 & -9+9-4 & 12-12+4 \\ 6-6+0 & -6+9-4 & 8-12+4 \\ 0-2+0 & 0+3-1 & 0-4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9-8+0 & -9+12-4 & 12-16+4 \\ 0-2+0 & 0+3+0 & 0-4+0 \\ -6+4+0 & 6-6+3 & -8+8-3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$A^3 = A^{-1}$

7). Show that the adjoint of $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ is $3A^T$

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Cofactor of -1 is =+ (1-4) = -3

Cofactor of -2 is =- (2+4) = -6

Cofactor of -2 is =+ (-4-2) = -6

Cofactor of 2 is =- (-2-4) = 6

Cofactor r of 1 is =+ (-4+1) = 3

cofactor of -2 is =- (2+4) = -6

cofactor of 2 is =+ (4+2) = 6

cofactor of -2 is =- (2+4) = -6

cofactor of 1 is =+ (-1+4) = 3

$[Aij] = \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}$

(Adj. A) =$[Aij]T$ =

$$\begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix} \text{.................................. (1)}$$

$$3A^T = 3\begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix} \text{.................. (2)}$$

There for (Adj. A) = $3A^T$ from (1) and (2).

8 .show that the adjoint of A = $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ is A itself .

$$A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$$

Cofactor of -4 is =+ (0-4) = -4

Cofactor of-3 is =- (3-4) = 1

Cofactor of -3 is =+ (4-0) = 4

Cofactor of 1 is =- (-9+12) = -3

Cofactor r of 0 is =+ (-12+12) = 0

cofactor of 1 is =- (-16+12) = 4

cofactor of 4 is =+ (-3+0) = -3

cofactor of 4 is =- (-4+3) = 1

cofactor of 3 is =+ (0+3) = 3

$[Aij]$ = $\begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}$

Adj. A = $(Aij)^T$ = $\begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}^T$

= $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ = A

9. If A =$\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$; P.T A^{-1} =A^T

A =$\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ =$\begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{2} \\ \frac{-2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{-2}{3} & \frac{2}{3} \end{bmatrix}$

$|A|$ = $\frac{1}{27}\begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix}$

= $\frac{1}{27}[12 + 12 + 3]$ =1

Cofactor of 3 is =+ (2+4) = 6

Cofactor of-3 is =- (-4-2) = 6

Cofactor of4 is =+ (4+-1) = 3

Cofactor of 2 is =- (4+2) = -6

cofactor of -3 is =+ (4-1) = 3

cofactor of 4 is =- (-4-2) = 6

cofactor of 0 is =+ (4-1) = 3

cofactor of -1 is =- (4+2) = -6

cofactor of 1 is =+ (4+2)= 6

$$[Aij] = \frac{1}{9}\begin{bmatrix} 6 & 6 & 3 \\ -6 & 3 & 6 \\ 3 & -6 & 6 \end{bmatrix}$$

$$(adj.\ A) = (Aij)T = \frac{1}{9}\begin{bmatrix} 6 & -6 & 3 \\ 6 & 3 & -6 \\ 3 & 6 & 6 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(adj\ .\ A) = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix} = A^{T}$$

10.For $A = \begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix}$, show that $A = A^{-1}$

$$A = \begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{vmatrix}$$

=-1(-15+16)-2(20-16) -2(-16+12)

= -1-8+8 = -1

Cofactor of-1 is =+ (-15+16) = 1

Cofactor of 2 is =- (20-16) = -4

Cofactor of-2 is =+ (-16+12) = -4

Cofactor of 4 is =- (10-8) = -2

cofactor of -3 is =+ (-5+8) = 3

cofactor of 4 is =- (4-8) = 4

cofactor of 4 is =+ (8-6) = 2

cofactor of -4 is =- (-4+8) = -4

cofactor of 5 is =+ (3-8) = -5

$$Aij = \begin{bmatrix} 1 & -4 & -4 \\ -2 & 3 & 4 \\ 2 & -4 & -5 \end{bmatrix}$$

$$(adj.\ A) = (Aij)T = \begin{bmatrix} 1 & -2 & 2 \\ -4 & 3 & -4 \\ -4 & 4 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(adj\ .\ A) = \frac{1}{-1}\begin{bmatrix} 1 & -2 & 2 \\ -4 & 3 & -4 \\ -4 & 4 & -5 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix}$$

Find the adjoint of matrices:

(i) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$; (ii) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{pmatrix}$;

Solution: (i) $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

the matrix of cofactor $[Aij] = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

Therefore $adjA = (Aij)^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(ii) $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$

cofactor of 1 is =+ (6-3) = 3

cofactor of 1 is =- (3+6) = -9

cofactor of 1 is =+ (-1-4) = -5

cofactor of 1 is =-(3+1) = -4

cofactor of 2 is =+(3-2) = 1

cofactor of -3 is =-(-1-2) = 3

cofactor of 2 is =+(-3-2) = -5

cofactor of -1 is =-(-3-1) = 4

cofactor of 3 is =+(2-1) = 1

$$A_{ij} = \begin{bmatrix}3 & -9 & -5\\ -4 & 1 & 3\\ -5 & 4 & 1\end{bmatrix}$$

There fore adj.A $=\begin{bmatrix}3 & -4 & -5\\ -9 & 1 & 4\\ -5 & 3 & 1\end{bmatrix}$

2.Find the adjoint of the matrix $A=\begin{bmatrix}-1 & 2\\ 1 & -4\end{bmatrix}$. and verify the result. $A(adj.A) = (adj.A)A = |A|I_2$

Solution $A = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}$. $A = \begin{vmatrix} -1 & 2 \\ 1 & -4 \end{vmatrix}$ = 4-2 = 2

the matrix of cofactor$[Aij]$=$\begin{bmatrix} -4 & -1 \\ -2 & -1 \end{bmatrix}$

There fore adjA=$(Aij)^T$ =$\begin{bmatrix} -4 & -2 \\ -1 & -1 \end{bmatrix}$

$$A(adj.A) = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} -4 & -2 \\ -1 & -1 \end{bmatrix}$$

$$= \quad = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A|I_2$$

$$(adj.A)A = \begin{bmatrix} -4 & -2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A|I_2$$

Hence $A(adj.A) = (adj.A)A = |A|I_2$

3.find the adjoint of matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ and verify the result.

$A(adj.A) = (adj.A)A = |A|I.$

Solution: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{vmatrix}$$

= 1(6-3) -1(3+6) +1(-1-4)

= 3-9-5 = -11

cofactor of 1 is =+ (6-3) = 3

cofactor of 1 is =- (3+6) = -9

cofactor of 1 is =+ (-1-4) = -5

cofactor of 1 is =-(3+1) = -4

cofactor of 2 is =+(3-2) = 1

cofactor of -3 is =-(-1-2) = 3

cofactor of 2 is =+(-3-2) = -5

cofactor of -1 is =-(-3-1) = 4

cofactor of 3 is =+(2-1) = 1

$$Aij = \begin{bmatrix} 3 & -9 & -5 \\ -4 & 1 & 3 \\ -5 & 4 & 1 \end{bmatrix}$$

There fore $adj.A = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$

$$A(adj.A) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix}$$

$$= -11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (-11)I_3 = |A|\, I_3$$

$$(adj.A)A = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= -11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (-11)I_3 = |\,A\,|\, I_3$$

Hence $A(adj.A) = (adj.A)A = |A|I_3$ Hence proved.

4. find the inverses of the following matrices : $\begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$A = \begin{vmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{vmatrix} = 2 \neq 0$$

cofactor of 3is =+ (2-0) = 2

cofactor of 1is =- (-2-0) = 2

cofactor of -1 is =+ (4+2) = 6

cofactor of 2 is =-(-1+2) = -1

cofactor of -2 is =+(-3+1) = -2

cofactor of 0 is =-(6-1) =-5

cofactor of 1 is =+(0-2) = -2

cofactor of 2 is =-(0+2) = -2

cofactor of -1 is =+(-6-2) = -8

$$Aij = \begin{bmatrix} 2 & 2 & 6 \\ -1 & -2 & -5 \\ -2 & -2 & -8 \end{bmatrix}$$

$$\text{There fore adj.A} = \begin{bmatrix} 2 & -1 & -2 \\ 2 & -2 & -2 \\ 6 & -5 & -8 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj .A} = \frac{1}{2}\begin{bmatrix} 2 & -1 & -2 \\ 2 & -2 & -2 \\ 6 & -5 & -8 \end{bmatrix}$$

5. if $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ verify that (i) $(AB)^{-1} = B^{-1} A^{-1}$

Solution: (i) $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 0+2 & -1+4 \\ 0+1 & -1+2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

To find A^{-1}

$|AB| = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 2-3 = -1$

$[Aij] = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$

$\text{adj } .A = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \text{adj } .A = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

To find B^{-1}

$|B| = \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = 0+1 = 1$

$[Bij] = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

$\text{adj } .B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

$B^{-1} = \frac{1}{|B|} \text{adj } .B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

To find $(AB)^{-1}$

$$|AB| = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 2-3 = 1$$

Matrix of cofactor of(AB) $= \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$

Therefore adj.(AB) $= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$

Therefore $(AB)^{-1} = \frac{1}{|AB|}$ (adj AB)$= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$

$$B^{-1}A^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

From (1) and) (2) $(AB)^{-1} = B^{-1}A^{-1}$

EXERCISE 1:2

Solve by matrix inversion method each of the following system of linear equations:

1.(i) 2x-y = 7, 3x-2y = 11

Solution: 2x-y = 7

3x-2y = 11

AX =B

$$A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 11 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} = -4+3 = -1$$

$X = A^{-1}B.$

$$(Aij) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$$

$$Adj.A = (Aij)^T = \begin{bmatrix} -2 & -1 \\ 3 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} (adj.A) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{pmatrix} 7 \\ 11 \end{pmatrix} = \begin{pmatrix} 14-11 \\ 21-22 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

X = 3 , y = -1.

1(ii). 7x+3y =-1, 2x+y= 0

Solution: 7x+3y =-1,

2x+y= 0

AX =B

$$\begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$A = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 7 & 3 \\ 2 & 1 \end{vmatrix} = 7-6 = 1$$

$X = A^{-1}B.$

$$(A_{ij}) = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}$$

$$Adj.A = (A_{ij})^T = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(adj.A) = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 + 0 \\ 2 + 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

X = -1, y =2.

2. x+y+z = 9 , 2x+5y+7z = 52 , 2x+y-z = 0.

Solution: x+y+z = 9

2x+5y+7z = 52

2x+y-z = 0

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 52 \\ 0 \end{pmatrix}$$

It is of the form AX =B ,

$X = A^{-1}B.$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix}$$

=1(-5-7)-1(-2-14)+1(2-10)

= -12+16-8 = -4

Cofactor of1 is =+ (-5-7) = -12

Cofactor of 1 is =- (-2-14) = 16

Cofactor of 1 is =+ (2-10) = -8

Cofactor of 2 is =- (-1-1) = 2

Cofactor of5 is =+ (-1-2) = -5

Cofactor of 7 is = - (1-2) = 1

Cofactor of 2 is =+ (7-5) = 2

Cofactor of 1 is =- (7-2) = -3

Cofactor of -1 is =+ (5-2) = 3

$$\text{Aij} = \begin{bmatrix} -12 & 16 & -8 \\ 2 & -5 & 1 \\ 2 & -3 & 3 \end{bmatrix}$$

$$(\text{adj. A}) = (Aij)T = \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj . A}) = \frac{1}{-4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$X = \frac{1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}\begin{pmatrix} 9 \\ 52 \\ 0 \end{pmatrix}$$

$$X = \frac{1}{4}\begin{bmatrix} 108 & -104 & 0 \\ -144 & 156 & 0 \\ 72 & -52 & 0 \end{bmatrix} = \frac{1}{4}\begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

X=1 , Y = 3, Z= 5

2. 2x-y+z= 7, 3x+y-5z = 13, x+y+z = 5

Solution: 2x-y+z= 7

3x+y-5z = 13

x+y+z = 5

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 13 \\ 5 \end{pmatrix}$$

It is of the form AX = B ,

$X = A^{-1}B.$

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{vmatrix}$$

=2(1+5) +1(3+5) +1(3-1)

= 12+8+2 = 22

Cofactor of 2is =+ (1+5) = 6

Cofactor of-1 is =- (3+5) = -8

Cofactor of 1 is =+ (3-1) = 2

Cofactor of 3 is =- (-1-1) = 2

Cofactor of 1 is =+ (2-1) = 1

Cofactor of -5 is =- (2+1) = -3

Cofactor of 1 is =+ (5-1) = 4

Cofactor of 1 is =- (-10-3) = 13

Cofactor of 1 is =+ (2+3) = 5

$$Aij = \begin{bmatrix} 6 & -8 & 2 \\ 2 & 1 & -3 \\ 4 & 13 & 5 \end{bmatrix}$$

$$(adj.\ A) = (Aij)T = \begin{bmatrix} 6 & 2 & 4 \\ -8 & 1 & 13 \\ 2 & -3 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(adj\ .\ A) = \frac{1}{22}\begin{bmatrix} 6 & 2 & 4 \\ -8 & 1 & 13 \\ 2 & -3 & 5 \end{bmatrix}$$

$$X = \frac{1}{22}\begin{bmatrix} 6 & 2 & 4 \\ -8 & 1 & 13 \\ 2 & -3 & 5 \end{bmatrix}\begin{pmatrix} 7 \\ 13 \\ 5 \end{pmatrix}$$

$$= \frac{1}{22}\begin{pmatrix} 42+26+20 \\ -56+13+65 \\ 14-39+25 \end{pmatrix}$$

$$= \frac{1}{22}\begin{pmatrix} 88 \\ 22 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

X = 4, y= 1, z= 0

5. x-3y-8z+10 = 0, 3x+y = 4, 2x+5y+6z = 13

Solution: x-3y-8z+10 = 0

3x+y = 4

2x+5y+6z = 13

$$\begin{bmatrix}1 & -3 & -8\\ 3 & 1 & 0\\ 2 & 5 & 6\end{bmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}-10\\ 4\\ 13\end{pmatrix}$$

It is of the form AX = B ,

X = A^{-1}B.

$$|A| = \begin{bmatrix}1 & -3 & -8\\ 3 & 1 & 0\\ 2 & 5 & 6\end{bmatrix}$$

$$A = \begin{vmatrix}1 & -3 & -8\\ 3 & 1 & 0\\ 2 & 5 & 6\end{vmatrix}$$

=1(6-0) +3(18-0) -8(15-2)

=6+54-104 = -44

Cofactor of1 is =+ (6-0) = 6

Cofactor of -3is =- (18-0) = -18

Cofactor of -8is =+ (15-2) = 13

Cofactor of3 is =- (-40+18) = -22

Cofactor of1 is =+ (6+16) = 22

Cofactor of 0 is =- (5+6) = -11

Cofactor of 2 is =+ (0+8) = 8

Cofactor of 5 is =- (0+24) = -24

Cofactor of 6 is =+ (1+9) = 10

$$\text{Aij} = \begin{bmatrix} 6 & -18 & 13 \\ -22 & 22 & -11 \\ 8 & -24 & 10 \end{bmatrix}$$

$$(\text{adj. A}) = (Aij)T = \begin{bmatrix} 6 & -22 & 8 \\ -18 & 22 & -24 \\ 13 & -11 & 10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj . A}) = \frac{1}{-44}\begin{bmatrix} 6 & -22 & 8 \\ -18 & 22 & -24 \\ 13 & -11 & 10 \end{bmatrix}$$

$$X = \frac{1}{-44}\begin{bmatrix} 6 & -22 & 8 \\ -18 & 22 & -24 \\ 13 & -11 & 10 \end{bmatrix}\begin{pmatrix} -10 \\ 4 \\ 13 \end{pmatrix}$$

$$= \frac{1}{-44}\begin{pmatrix} -44 \\ -44 \\ -44 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

X =1 , y= 1, z= 1.

Solve by matrix inversion method each of the following system of linear equations: x+y = 3, 2x+3y = 8

Solution : x+y = 3

$2x+3y = 8$

$AX = B$

$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 11 \end{pmatrix}$

$|A| = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3-2 = 1$

$X = A^{-1}B.$

$(A_{ij}) = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

$Adj.A = (A_{ij})^T = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}(adj.A) = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

$\begin{matrix} x \\ y \end{matrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{pmatrix} 7 \\ 11 \end{pmatrix} = \begin{pmatrix} 14-11 \\ 21-22 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$x = 1 , y = 2$

2. $2x-y+3z = 9$, $x+y+z = 6$, $x-y+z = 2$.

Solution : $2x-y+3z = 9$

$x+y+z = 6$

$x-y+z = 2.$

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 2 \end{pmatrix}$$

It is of the form AX =B ,

$X = A^{-1}B.$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

=2(1+1)+1(1-1)+3(-1-1)

= 4+0-6 = -2 $\neq$ 0

Cofactor of 2 is =+ (1+1) = 2

Cofactor of -1 is =- (1-1) = 0

Cofactor of 3 is =+ (-1-1) = -2

Cofactor of 1 is =- (-1+3) = -2

Cofactor of 1 is =+ (2-3) = -1

Cofactor of 1 is =- (-2+1) = 1

Cofactor of 1 is =+ (-1-3) = -4

Cofactor of -1 is =- (2-3) = 1

Cofactor of 1 is =+ (2+1) = 3

$$A_{ij} = \begin{bmatrix} 2 & 0 & -2 \\ -2 & -1 & 1 \\ -4 & 1 & 3 \end{bmatrix}$$

$$(\text{adj. } A) = (Aij)T = \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj . } A) = \frac{1}{-2}\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-2}\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}\begin{pmatrix} 9 \\ 6 \\ 2 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

x=1 , y= 2, z = 3

RANK OF MATRIX

Find the rank of the following matrices:

1. $\begin{bmatrix} 1 & 1 & -1 \\ 3 & -2 & 3 \\ 2 & -3 & 4 \end{bmatrix}$

Solution: $A=\begin{bmatrix}1 & 1 & -1\\ 3 & -2 & 3\\ 2 & -3 & 4\end{bmatrix}$

$R2 \to R_2-3R_1$; $R_3 \to R_3-2R_1$

$$\sim\begin{bmatrix}1 & 1 & -1\\ 0 & -5 & 6\\ 0 & -5 & 6\end{bmatrix}$$

$R_3 \to R_3-R_2$

$$\sim\begin{bmatrix}1 & 1 & -1\\ 0 & -5 & 6\\ 0 & 0 & 0\end{bmatrix}$$

The last equivalent matrix is in the echelon form . it has two non zero rows .

Therefore p(A) = 2

2). $\begin{bmatrix}6 & 12 & 6\\ 1 & 2 & 1\\ 4 & 8 & 4\end{bmatrix}$

Solution: $A=\begin{bmatrix}6 & 12 & 6\\ 1 & 2 & 1\\ 4 & 8 & 4\end{bmatrix}$

$R1 \to \frac{1}{6}R1$; $R_3 \to \frac{1}{4}R3$

$$\sim\begin{bmatrix}1 & 2 & 1\\ 1 & 2 & 1\\ 1 & 2 & 1\end{bmatrix}$$

$R2 \to R_2-R_1$; $R_3 \to R_3-R_1$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

The last equivalent matrix is in the echelon form . it has two non zero rows .

Therefore p(A) = 1

5. $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 4 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

let A= $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 4 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{pmatrix} R_2 \to R_2 - 2R_1 \ , R_3 \to R_3 - 3R_1$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} R_3 \to R_3 - 2R_2$$

The last equivalent matrix is in the echelon form . it has two non zero rows . Therefore p(A) = 1

6. $\begin{pmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & -6 \end{pmatrix}$

let A= $\begin{pmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & -6 \end{pmatrix}$

$R_2 \to R_2 + 2R_1; \ R_3 \to R_3 + R_1$

$$\sim \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 10 & 10 \end{pmatrix} \quad R_3 \rightarrow R_3\text{-}2R_2$$

$$\sim \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The last equivalent matrix is in the echelon form. it has two non zero rows . Therefore p (A) = 2.

1 find the rank of the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

Solution: $A=\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

$R2 \rightarrow R_2\text{-}2R_1$; $R_3 \rightarrow R_3\text{-}3R_1$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

The last equivalent matrix is in the echelon form . it has two non zero rows is 2.

Therefore p(A) = 2

2 . find the rank of the matrix: $\begin{pmatrix} 1 & 2 & 3 & -1 \\ 2 & 4 & 6 & -2 \\ 3 & 6 & 9 & -3 \end{pmatrix}$

$$\text{solution : } A = \begin{pmatrix} 1 & 2 & 3 & -1 \\ 2 & 4 & 6 & -2 \\ 3 & 6 & 9 & -3 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1 \qquad R_3 \to R_3 - 2R_1$

$$\sim \begin{pmatrix} 1 & 2 & 3 & -1 \\ 2 & 4 & 6 & -2 \\ 3 & 6 & 9 & -3 \end{pmatrix}$$

The last equivalent matrix is in the echelon form . it has one non zero rows .

Therefore p(A) = 1

3 . find the rank of the matrix: $\begin{pmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{pmatrix}$

$$\text{solution : } A = \begin{pmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 4 & 3 & 6 & 7 \\ 0 & 1 & 2 & 1 \end{pmatrix} \quad \sim C1 \leftrightarrow C3$$

$R_2 \to R_2 - R_1$

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & -5 & -10 & -5 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

$R_2 \longrightarrow \frac{1}{-5}$

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

$R_3 \longrightarrow R_3 - R_2$

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The last equivalent matrix is in the echelon form . it has two non zero rows .

Therefore p(A) = 2

1.4 (1) (Cramer's rule method) ⟶ (Determinant Method)

Consider the system of non homogeneous equations of

$a_{11}x+a_{12}y = b_1$

$a_{21}x+a_{22}y = b_2$

let $\Delta = \begin{pmatrix} a11 & a12 \\ a21 & a22 \end{pmatrix}$

$\Delta x = \begin{pmatrix} b1 & a12 \\ b2 & a22 \end{pmatrix}$

$\Delta y = \begin{pmatrix} a11 & b1 \\ a12 & b2 \end{pmatrix}$

Then $x = \frac{\Delta x}{\Delta}$; $y= \frac{\Delta y}{\Delta}$ find x =value and y= value

Example: solve the following non homogeneous system of linear equations by determinant method.

1. 3x+2y = 5; x+3y = 4

Solution: 3x+2y = 5

x+3y = 4

$\Delta = \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$

= 9-2 = 7

$\Delta x = \begin{vmatrix} 5 & 2 \\ 4 & 3 \end{vmatrix}$

= 15-8 = 7

$\Delta y = \begin{vmatrix} 3 & 5 \\ 1 & 4 \end{vmatrix}$

= 12-5 = 7

Then x = $\frac{\Delta x}{\Delta}$ = $\frac{7}{7}$ =1

y= $\frac{\Delta y}{\Delta}$ =$\frac{7}{7}$ = 1

find x =1 and y= 1

2. 2x+3y = 5; 4x+6y = 12

Solution: 2x+3y = 5

4x+6y = 12

$$\Delta = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix}$$

=12-12 = 0

$$\Delta x = \begin{vmatrix} 5 & 3 \\ 12 & 6 \end{vmatrix}$$

= 30-36 = -6 =0

$$\Delta y = \begin{vmatrix} 2 & 5 \\ 4 & 12 \end{vmatrix}$$

= 24-20 = 4 = 0

Since Δ = 0 ; Δ ≠ 0 and the system is inconsistent.

3. `4x+5y = 9; 8x+10y = 18

Solution: $4x+5y = 9$

$8x+10y = 18$

$$\Delta = \begin{vmatrix} 4 & 5 \\ 8 & 10 \end{vmatrix}$$

$= 40-40 = 0$

$$\Delta x = \begin{vmatrix} 9 & 5 \\ 18 & 10 \end{vmatrix}$$

$= 90-90 = 0$

$$\Delta y = \begin{vmatrix} 4 & 9 \\ 8 & 18 \end{vmatrix}$$

$=72-72 = 0$

Since $\Delta = \Delta x = \Delta y = 0$

And at least one of the coefficients is non zero the system is consistent and has many solutions.

Let y = k .then $x=\frac{9-5k}{4}$.

Therefore the solution set is $(x,y) = (\frac{9-5k}{4}, k)$ where k€R

4. $X+Y+Z = 4$; $X-Y+Z = 2$; $2X+Y-Z = 1$

Solution: $X+Y+Z = 4$

$X-Y+Z = 2$

$2X+Y-Z = 1$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

= 1(1-1)-1(-1-2)+1(1+2)

= 0+3+3 = 6

$$\Delta x = \begin{vmatrix} 4 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

= 4(1-1)-1(-2-1)+1(2+1)

= 0+3+3 = 6

$$\Delta y = \begin{vmatrix} 1 & 4 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

= -3+12-3 = 6

$$\Delta z = \begin{vmatrix} 1 & 1 & 4 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix}$$

= 1(-1-2)-1(1-4)+4 (1+2)

= -3+3+12 = 12

Then x = $\frac{\Delta x}{\Delta}$ = $\frac{6}{6}$ =1

$y = \frac{\Delta y}{\Delta} = \frac{6}{6} = 1$

$z = \frac{\Delta z}{\Delta} = \frac{12}{6} = 2$

5. 2X+Y-Z = 4; X+Y-2Z = 0; 3X+2Y-3Z = 4

Solution: 2X+Y-Z = 4

X+Y-2Z = 0

3X+2Y-3Z = 4

$$\Delta = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ 3 & 2 & -3 \end{vmatrix}$$

= 2(-3+4)-1(-3+6)-1(2-3)

= 2-3+1 = 0

$$\Delta x = \begin{vmatrix} 4 & 1 & -1 \\ 0 & 1 & -2 \\ 4 & 2 & -3 \end{vmatrix}$$

= 4(-3+4)-1(0+8)-1(0-4)

= 4-8+4 = 0

$$\Delta y = \begin{vmatrix} 2 & 4 & -1 \\ 1 & 0 & -2 \\ 3 & 4 & -3 \end{vmatrix}$$

= 2(0+8)-4(-3+6)-1(4-0)

= 16-12-4 = 0

$$\Delta z = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 1 & 0 \\ 3 & 2 & 4 \end{vmatrix}$$

= 2(4-0)-1(4-0)+ 4(2-3)

= 8-4-4 = 0

Since Δ =Δx = Δy = Δz =0 .the system is consistent and has many solution .also all

2x2 minor of Δ≠0. The system is reduced to equation.

Let z = k

2x+y-k = 4   2x+y = 4+k

X+ y-2k = 0   x+ y = 2k

$$\Delta = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

= 2-1 = 1

$$\Delta x = \begin{vmatrix} 4+k & 1 \\ 2k & 1 \end{vmatrix}$$

$= 4+k-2k = 4-k$

$$\Delta y = \begin{vmatrix} 2 & 4+k \\ 1 & 2k \end{vmatrix}$$

$=4k-4-k = 3k-4$

Then $x = \frac{\Delta x}{\Delta} = \frac{4-k}{1} = 4-k$

$= \frac{\Delta y}{\Delta} = \frac{3k-4}{1} = 3k-4$

x =4-k and y= 3k-4 and z=k

solution set is (4-k,3k-4,k) where k €R

6. 3x+y-z = 2 ; 2x-y+2z =6 ; 2x+y-2z = -2

Solution: 3x+y-z = 2

2x-y+2z =6

2x+y-2z = -2

$$\Delta = \begin{vmatrix} 3 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & 1 & -2 \end{vmatrix}$$

=3(2-2)-1(-4-4)-1(2+2)

= 0+8-4 = 4

$$\Delta x = \begin{vmatrix} 2 & 1 & -1 \\ 6 & -1 & 2 \\ -2 & 1 & -2 \end{vmatrix}$$

=2(2-2)-1(-12+4)-1(6-2)

= 0+8-4 = 4

$$\Delta y = \begin{vmatrix} 3 & 2 & -1 \\ 2 & 6 & 2 \\ 2 & -2 & -2 \end{vmatrix}$$

=-24+16+8 = 8

$$\Delta z = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -1 & 6 \\ 2 & 1 & -2 \end{vmatrix}$$

= 3(2-6)-1(-4-12) + 2(2+2)

=-12+16+8 = 12

Then $x = \frac{\Delta x}{\Delta} = \frac{4}{4} = 1$

$y = \frac{\Delta y}{\Delta} = \frac{8}{4} = 2$

$z = \frac{\Delta z}{\Delta} = \frac{12}{4} = 3$

7. X+2y+z = 6; 3x+3y-z = 3; 2x+y-2z = -3

Solution: X+2y+z = 6

3x+3y-z = 3

2x+y-2z = -3

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 3 & -1 \\ 2 & 1 & -3 \end{vmatrix}$$

= 1(-6+1)-2(-6+2)+1(3-6)

=-5+8-3 = 0

$$\Delta x = \begin{vmatrix} 6 & 2 & 1 \\ 3 & 3 & -1 \\ -3 & 1 & -2 \end{vmatrix}$$

= 6(-6+1)-2(-6-3)+ 1(3+9)

= -30+18+12 = 0

$$\Delta y = \begin{vmatrix} 1 & 6 & 1 \\ 3 & 3 & -1 \\ 2 & -3 & -2 \end{vmatrix}$$

= 1(-6-3)-6(-6+2)+ 1(-9-6)

=-9+24-15 = 0

$$\Delta z = \begin{vmatrix} 1 & 2 & 6 \\ 3 & 3 & 3 \\ 2 & 1 & -3 \end{vmatrix}$$

= 1(-9-3)-2(-9-6) + 6(3-6)

=-12+30-18 = 0

Since Δ =Δx = Δy = Δz =0 .the system is consistent and has many solution .also all

2x2 minor of Δ≠0. The system is reduced to equation.

Let z = k

x+2y+k = 6   x+2y = 6-k

3X+3y-k = 3   3x+3y = 3+k

$$\Delta = \begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix}$$

= 3-6 = -3

$$\Delta x = \begin{vmatrix} 6-k & 2 \\ 3+k & 3 \end{vmatrix}$$

= 18-3k-6-2k = 12-5k

$$\Delta y = \begin{vmatrix} 1 & 6-k \\ 3 & 3+k \end{vmatrix}$$

=3+k-18+3k= 4k-15

Then x = $\frac{\Delta x}{\Delta} = \frac{12-5k}{-3} = \frac{5k-12}{3}$

= $\frac{\Delta y}{\Delta} = \frac{4k-15}{-3} = \frac{15-4k}{3}$

x = $\frac{5k-12}{3}$ and y= $\frac{15-4k}{3}$ and z=k

solution set is ($\frac{5k-12}{3}$,$\frac{15-4k}{3}$,k) where k €R.

8. 2x –y+z = 2 ; 6x -3y +3z = 6; 4x-2y +2z = 4

solution : 2x –y+z = 2

6x -3y +3z = 6

4x-2y +2z = 4

$$\Delta = \begin{vmatrix} 2 & -1 & 1 \\ 6 & -3 & 3 \\ 4 & -2 & 2 \end{vmatrix}$$

= 2(-6+6)+1(12-12)+1(-12+12)

= 0

$$\Delta x = \begin{vmatrix} 2 & -1 & 1 \\ 6 & -3 & 3 \\ 4 & -2 & 2 \end{vmatrix}$$

= 2(-6+6)+1(12-12)+1(-12+12)

= 0

$$\Delta y = \begin{vmatrix} 2 & 2 & 1 \\ 6 & 6 & 3 \\ 4 & 4 & 2 \end{vmatrix}$$

= 2(-12+12)-2(12-12)+1(24+24)

= 0

$$\Delta z = \begin{vmatrix} 2 & -1 & 2 \\ 6 & -3 & 6 \\ 4 & -2 & 4 \end{vmatrix}$$

= 2(-12+12)+1(24-24)+1(-12+12)

= 0

$\Delta = \Delta x = \Delta y = \Delta z = 0$

all (2x2) minor are also zeros . but atleast one of Aij in Δ is non zero.

the system is consistent and has many solution . all the three equation reduce to one solution . 2x-y +z = 2

put z= k then 2x –y = 2-k

let y = s , then x = $(\frac{2-k+s}{2}, s, k)$ where s, k ∈ R

9. $\frac{1}{x}+\frac{2}{y}-\frac{1}{z}=1$; $\frac{2}{x}+\frac{4}{y}+\frac{1}{z}=5$; $\frac{3}{x}-\frac{2}{y}-\frac{2}{z}$

solution : let $\frac{1}{x}=a$; $\frac{1}{y}=b$; $\frac{1}{z}=c$.

a+ 2b –c = 1

2a+ 4b +c = 5

3a- 2b –2c = 0

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ 3 & -2 & -2 \end{vmatrix}$$

= 1(-8+2)-2(-4-3)-1(-4-12)

= -6+14+16 = 24

$$\Delta a = \begin{vmatrix} 1 & 2 & -1 \\ 5 & 4 & 1 \\ 0 & -2 & -2 \end{vmatrix}$$

=1 (-8+2)-2(-10-0)-1(-10-0)

= -6+20+10= 24

$$\Delta b = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 5 & 1 \\ 3 & 0 & -2 \end{vmatrix}$$

= 1 (-10-0)-1(-4-3)-1(0-15)

= -10+7+15 = 12

$$\Delta c = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & -2 & 0 \end{vmatrix}$$

= 1(0+10)-2(0-15)+1(-4-12)

= 10+30-16 = 24

a $= \frac{\Delta a}{\Delta} = \frac{24}{24} = 1 => \frac{1}{x} = 1 => x = 1$

b $= \frac{\Delta b}{\Delta} = \frac{12}{24} = \frac{1}{2} => \frac{1}{y} = \frac{1}{2} => y = 2$

c $= \frac{\Delta c}{\Delta} = \frac{24}{24} = 1 => \frac{1}{z} = 1 => z = 1$

10. a small seminar hall hold 100 chars . three different colours (red , blue , and green) of chairs are available . the cost of red chairs is Rs . 240 , cost of the blue chairs is Rs 260 the cost of the green chairs is Rs . 300 . the total cost of the chairs if Rs . 25,000 . find atleast 3 different solution of the number of chairs in each colour to be purchased .

solution : let x ,y ,z be the no. of red , blue , green chairs .

given that x + y +z = 100

240x +x260y+300z = 25000

$\div 20$

12x+13y+15z = 1250

x + y =100 – k

x + y = 1250 -15 k

$\Delta = \begin{vmatrix} 1 & 1 \\ 12 & 13 \end{vmatrix}$ = 13-12 = 1

$$\Delta x = \begin{vmatrix} 100-k & 1 \\ 1250-15k & 13 \end{vmatrix}$$

= 1300 -13k-1250+15k

= 50 +2k

$$\Delta y = \begin{vmatrix} 1 & 100-k \\ 12 & 1250-15k \end{vmatrix}$$

= 1250 -15k-1200+12k

= 50 -3k

x $= \frac{\Delta x}{\Delta} = \frac{50+2k}{1} = 50+2k$

y $= \frac{\Delta y}{\Delta} = \frac{50-3k}{1} = 50-3k$

z = k

the solution set is (50+2k ,50-3k ,k) where s, k $\in R$.

. x+2y+z = 7 ; 2x-y+2z =4 ; x+y-2z = -1

Solution: x+2y+z = 7

2x-y+2z =4

x+y-2z = -1

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & -2 \end{vmatrix} = 15$$

$$\Delta x = \begin{vmatrix} 7 & 2 & 1 \\ 4 & -1 & 2 \\ -1 & 1 & -2 \end{vmatrix} = 15$$

$$\Delta y = \begin{vmatrix} 1 & 7 & 1 \\ 2 & 4 & 2 \\ 1 & -1 & -2 \end{vmatrix} = 30$$

$$\Delta z = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & -2 \end{vmatrix} = 30$$

Then $x = \frac{\Delta x}{\Delta} = \frac{15}{15} = 1$

$y = \frac{\Delta y}{\Delta} = \frac{30}{15} = 2$

$z = \frac{\Delta z}{\Delta} = \frac{30}{15} = 2$

solution is (x ,y , z) = (1 ,2 ,2)

. x+y+2z = 6; 3x+y-z = 2; 4x+2y+z = 8

Solution : x+y+2z = 6

3x+y-z = 2

4x+2y+z = 8

$$\Delta = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 1 & -1 \\ 4 & 2 & 1 \end{vmatrix} = 0$$

$$\Delta x = \begin{vmatrix} 6 & 1 & 2 \\ 2 & 1 & -1 \\ 8 & 2 & 1 \end{vmatrix} = 0$$

$$\Delta y = \begin{vmatrix} 1 & 6 & 2 \\ 3 & 2 & -1 \\ 4 & 8 & 1 \end{vmatrix}$$

= 0

$$\Delta z = \begin{vmatrix} 1 & 1 & 6 \\ 3 & 1 & 2 \\ 4 & 2 & 8 \end{vmatrix}$$

= 0

Since Δ =Δx = Δy = Δz =0 .the system is consistent and has many solution .also all

2x2 minor of Δ≠0. The system is reduced to equation.

Let z = k

x+y+2k = 6   x+y = 6-2k

3x+y-k = 2   3x+y = 2+k

$$\Delta = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}$$

= 1-3 = -2

$$\Delta x = \begin{vmatrix} 6-2k & 1 \\ 2+k & 1 \end{vmatrix}$$

= 6-2k-2-k = 4-3k

$$\Delta y = \begin{vmatrix} 1 & 6-2k \\ 3 & 2+k \end{vmatrix}$$

= 2+k-18+16k = 7k-16

Then x = $\frac{\Delta x}{\Delta} = \frac{4-3k}{-2} = \frac{3k-4}{2}$

$= \frac{\Delta y}{\Delta} = \frac{7k-16}{-2} = \frac{16-7k}{2}$

x = $\frac{3k-4}{2}$ and y= $\frac{16-7k}{2}$ and z=k

solution set is($\frac{3k-4}{2}$,$\frac{16-7k}{2}$,k) where k €R.

. x +y+2z = 4 ; 2x +2y +4z = 8; 3x+3y +6z = 12

solution : x +y+2z = 4

2x +2y +4z = 8

3x+3y +6z = 12

$$\Delta = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{vmatrix}$$

$= 0$

$$\Delta x = \begin{vmatrix} 4 & 1 & 2 \\ 8 & 2 & 4 \\ 12 & 3 & 6 \end{vmatrix}$$

$= 0$

$$\Delta y = \begin{vmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ 3 & 12 & 6 \end{vmatrix}$$

$= 0$

$$\Delta z = \begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 8 \\ 3 & 3 & 12 \end{vmatrix}$$

$= 2(-12+12)+1(24-24)+1(-12+12)$

$= 0$

$\Delta = \Delta x = \Delta y = \Delta z = 0$

all (2x2) minor are also zeros . but atleast one of Aij in Δ is non zero.

the system is consistent and has many solution . all the three equation reduce to one solution . $x+y+2z = 4$

put $x= s$ then $s+t+2z = 4 \Rightarrow z = \frac{4-s-t}{2}$

let $y = t$, then $x = (s, t, \frac{4-s-t}{2})$ where s, k $\in R$

. A bag contain 3 types of coins namely Re. 1 ,Re. 2 , Re. 5 .there are 30 coins amounting to Re. 100 in total . find the number of coins in each category .

solution : let x ,y ,z be the no. of coins in each Re. 1 ,Re. 2 , Re. 5.

given that x + y +z = 30

x +2y+5z = 100

x + y = 30 – k

x + y = 100-5 k

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2-1 = 1$$

$$\Delta x = \begin{vmatrix} 30-k & 1 \\ 100-5k & 2 \end{vmatrix}$$

= 2(30-k) -(100-5k)

= 3k-40

$$\Delta y = \begin{vmatrix} 1 & 30-k \\ 1 & 100-5k \end{vmatrix}$$

= (100-5k) –(30-k)

= 70 -4k

$$x = \frac{\Delta x}{\Delta} = \frac{3k-40}{1} = 3k-40$$

$$y = \frac{\Delta y}{\Delta} = \frac{70-4k}{1} = 70-4k$$

z = k

the solution set is(x , y z) = (3k − 40 ,70 -4k,k) where s, k ∈ R.

Since the number of coins is a non – negative integer , k = 0 ,1 ,3 ….

Moreover 3k – 40 ≥ 0 , 70- 4k≥0 , = > 14 ≤x≤ 17

The possible solution are (2,14,14)(5,10,15)(8,6,16)(11,2,17).

EXERCISE : 1. 5

examine the consistency of the following of the equations . if it is consistent then solve the sums .(using by rank method)

4x + 3y + 6z = 25 ; x+ 5y + 7z = 13 ; 2x +9y + z = 1

solution : 4x + 3y + 6z = 25

x+ 5y + 7z = 13

2x +9y + z = 1

$$A = \begin{bmatrix} 4 & 3 & 6 \\ 1 & 5 & 7 \\ 2 & 9 & 1 \end{bmatrix}$$

$$(A,B) = \begin{bmatrix} 4 & 3 & 625 \\ 1 & 5 & 713 \\ 2 & 9 & 1\ 1 \end{bmatrix} \sim$$

$$(A,B) = \begin{bmatrix} 1 & 5 & 713 \\ 4 & 3 & 625 \\ 2 & 9 & 1\ 1 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$R_2 \to R_2 - 4R_1$; $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 5 & 7 & 13 \\ 0 & -17 & -22 & -27 \\ 0 & -1 & -13 & -25 \end{bmatrix}$$

$R_2 \to (-R_2)$; $R_3 \to (-R_3)$

$$\sim \begin{bmatrix} 1 & 5 & 7 & 13 \\ 0 & 17 & 22 & 27 \\ 0 & 1 & 13 & 25 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 5 & 7 & 13 \\ 0 & 1 & 13 & 25 \\ 0 & 17 & 22 & 27 \end{bmatrix}$$

$R_3 \to R_3 - 17R_1$

$$\sim \begin{bmatrix} 1 & 5 & 7 & 13 \\ 0 & 1 & 13 & 25 \\ 0 & 0 & -199 & -398 \end{bmatrix}$$

$=> \rho(A,B) = 3$ and also $\rho(A) = 3$ = no . of unknowns

hence the system is consistent and has unique solution.

$-199z = -398$ $\quad$ $y + 13z = 25$ $\quad$ $x + 5y + 7z = 13$

$z = 2$ $\quad$ $y + 26 = 25$ $\quad$ $x - 5 + 14 = 13$

$y = -1$

solution is $x = 4$, $y = -1$, $z = 2$

(ii) x -3y -8z = -10 ; 3x + y -4z = 0 ; 2x + 5y +6z -13 = 0

solution : x -3y -8z = -10

3x + y -4z = 0

2x + 5y +6z -13 = 0

$$A = \begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & -4 \\ 2 & 5 & 6 \end{bmatrix}$$

$$(A, B) = \begin{bmatrix} 1 & -3 & -8 & -10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 13 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$; $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 11 & 22 & 33 \end{bmatrix}$$

$R_2 \to (R_2 \div 10)$; $R_3 \to (-R_3 \div 11)$

$$\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

= > $\rho(A,B)$ = 2 and also $\rho(A) \neq$ no . of unknowns .

hence the system is consistent and has unique solution.

let z = k

$x - 3y = -10 + 8k$

$\Rightarrow 3x - 9y = -30 + 24k$

$3x - 9y = -30 + 24k$

$3x + y = 4k$

(-) (-) (-)

$-10y = -30 + 20k$

$y = -2k + 3$

$x = -10 + 8k + 3(-2k + 3)$

$x = 2k - 1$

The solution set is (2k -1 , -2k +3 ,k) , where $k \in R$.

(iii). $x + y + z = 7$; $x + 2y + 3z = 18$; $y + 2z = 6$.

solution : $x + y + z = 7$

$x + 2y + 3z = 18$

$y + 2z = 6$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$(A, B) = \begin{bmatrix} 1 & 1 & 1\ 7 \\ 1 & 2 & 3 18 \\ 0 & 1 & 2\ 6 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$$\sim\begin{bmatrix}1 & 1 & 1\ 7\\ 0 & 1 & 2 11\\ 0 & 0 & 2\ 6\end{bmatrix}\ R_3 \rightarrow R_3 - R_2$$

$$\sim\begin{bmatrix}1 & 1 & 1\ 7\\ 0 & 1 & 2 11\\ 0 & 0 & 0 -5\end{bmatrix}$$

$=> \rho(A,B) = 3$ and also $\rho(A) = 2$.

hence the system is inconsistent and has no solution.

(iv) .x - 4y + 7z = 14 ; 3x + 8y- 2z = 13 ; 7x -8y +26z = 5

solution : x - 4y + 7z = 14

3x + 8y- 2z = 13

7x -8y +26z = 5

$$A = \begin{bmatrix}1 & -4 & 7\\ 3 & 8 & -2\\ 7 & -8 & 26\end{bmatrix}$$

$$(A,B) = \begin{bmatrix}1 & -4 & 7\ 14\\ 3 & 8 & -2 13\\ 7 & -8 & 26\ 5\end{bmatrix}$$

$R_2 \rightarrow R_2 - 3R_1$ $R_3 \rightarrow R_3 - 7R_1$

$$\sim\begin{bmatrix}1 & -4 & 7 & 14\\ 0 & 20 & -2\ 3 & -29\\ 0 & 20 & -23 & -93\end{bmatrix}\ R_3 - R_2$$

$$\sim\begin{bmatrix}1 & -4 & 7 & 14\\ 0 & 20 & -23 & -29\\ 0 & 0 & 0 & -64\end{bmatrix}\ R_3 \rightarrow R_3 - R_1$$

$=> \rho(A,B) = 3$ and also $\rho(A) = 2$.

hence the system is inconsistent and has no solution.

(V) $X + Y - Z = 1$; $2X + 2Y - 2Z = 2$; $-3X - 3Y + 3Z = -3$

solution : $X + Y - Z = 1$

$2X + 2Y - 2Z = 2$ => dividing by 2

$-3X - 3Y + 3Z = -3$ => dividing by -3

all three equation are one and the same .

there is only one equation in three unknowns .

hence the system is consistent but has many solution .

let $z = k_2$; $y = k_1$ then

$x + y - z = 1$

$x = 1 - k_1 + k_2$

$x = (1 - k_1 + k_2 , k_1 , k_2)$ $k_1 , k_2 \in R$.

2. discuss the solution of the system of equation for all values of λ

$x + y + z = 2$; $2x + y - 2z = 2$; $\lambda x + y + 4z = 2$

solution : $x + y + z = 2$

$2x + y - 2z = 2$

$\lambda x + y + 4z = 2$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ \lambda & 1 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ \lambda & 1 & 4 \end{vmatrix}$$

$= 1(4+2)-1(8+2\lambda)+1(2-\lambda)$

$= 6-8-2\lambda+2-\lambda = -3\lambda$

where $\lambda \neq 0$ $|A| \neq 0$ => the system has unique solution .

let $\lambda = 0$. then $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$

$$(A,B) \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & -2 & 2 \\ \lambda & 1 & 4 & 2 \end{bmatrix}$$

$$(A,B) \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & -1 & -4 & -2 \\ 0 & 1 & 4 & 2 \end{bmatrix} \quad R2 \rightarrow R_2 - 2R_1$$

$R_2 \rightarrow R_2 X(-1)$; $R_3 \rightarrow R_2 + R_3$

$$(A,B) \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

=> $\rho(A,B) = 2$ and also $\rho(A) = 2 \neq$ no .of unknowns.

hence the system is consistent and has many solution.

let z = k

x + y = 2 – k

2x + y = 2+2k

-x = -3 k

$x = 3k$

hence $y = 2-4k$

Therefore solution is $(3k, 2-4k, k)$, $k \in R$.

3.for what value of k , the system of equations. $kx + y + z = 1$; $x + ky + z = 1$;

$x + y + kz = 1$ have (i) unique solution , (ii) more then one solution and (iii) no solution.

solution : $kx + y + z = 1$

$x + ky + z = 1$

$x + y + kz = 1$

$$A = \begin{bmatrix} K & 1 & 1 \\ 1 & K & 1 \\ 1 & 1 & K \end{bmatrix} \quad ; \quad (A, B) = \begin{bmatrix} K & 1 & 1 & 1 \\ 1 & K & 1 & 1 \\ 1 & 1 & K & 1 \end{bmatrix}$$

$|A| = K(K^2 - 1) - 1(K-1) + 1(1-K)$

$= K(K^2 - 1) - 1(K-1) - 1(K-1)$

$= (K-1)(K(K+1) - 1 - 1)$

$= (K-1)(K^2 + K - 2)$

$= (K-1)(K+2)(K-1) = (K-1)^2 (K+2)$

$\Rightarrow (K-1)^2 (K+2) = 0$ then $k = 1, -2$

suppose $k \neq 1$ and $k \neq -2$ then $|A| \neq 0$

=> the system is consistent and has unique solution .

(ii) let $k = 1$. then the system reduces to a single equation

$x + y + z = 1$

the system will have many solution .

(iii) let $k = -2$

$$(A,B) = \begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix}$$

$$(A,B) \sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ -2 & 1 & 1 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$R_2 \to R_2 + 2R_1$; $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & -3 & 3 & 3 \\ 0 & 3 & -3 & 0 \end{bmatrix}$$

$R_2 \to \frac{R2}{3}$; $R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$=> \rho(A,B) = 3$ and also $\rho(A) = 2$

hence the system is inconsistent and has no solution.

vector algebra

1. Find $\vec{a} \cdot \vec{b}$ when $\vec{a} = 2\vec{i} + 2\vec{j} - \vec{k}$ and $b = 6\vec{i} - 3\vec{j} + 2\vec{k}$

Solution : $\vec{a} = 2\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = 6\vec{i} - 3\vec{j} + 2\vec{k}$

$$\vec{a} \cdot \vec{b} = (2)(6) + (2)(-3) + (-1)(2)$$

$$= 12 - 6 - 2 = 4$$

2. If $\vec{a} = \vec{i} + \vec{j} + 2\vec{k}$ and $\vec{b} = 3\vec{i} + 2\vec{j} - \vec{k}$ find $(\vec{a} + 3\vec{b}) \cdot (2\vec{a} - \vec{b})$

Solution

$$\vec{a} = \vec{i} + \vec{j} + 2\vec{k}, \quad \vec{b} = 3\vec{i} + 2\vec{j} - \vec{k}$$

$$\vec{a} + 3\vec{b} = (\vec{i} + \vec{j} + 2\vec{k}) + 3(3\vec{i} + 2\vec{j} - \vec{k})$$

$$= (\vec{i} + \vec{j} + 2\vec{k}) + (9\vec{i} + 6\vec{j} - 3\vec{k})$$

$$= (10\vec{i} + 7\vec{j} - \vec{k})$$

$$2\vec{a} - \vec{b} = 2(\vec{i} + \vec{j} + 2\vec{k}) - (3\vec{i} + 2\vec{j} - \vec{k})$$

$$(\vec{a} + 3\vec{b})(2\vec{a} - \vec{b}) = (10\vec{i} + 7\vec{i} - \vec{k}) \cdot (-\vec{i} + 5\vec{k})$$

$$= -10 - 5 = -15$$

3. find λ so that the vectors $2i + \lambda\vec{j} + \vec{k}$ and $\vec{i} - 2\vec{j} + \vec{k}$ are perpendicular to each other.

Solution: Let $\vec{a} = 2\vec{i} + \lambda\vec{j} + \vec{k}$

$$\vec{b} = \vec{i} - 2\vec{j} + \vec{k}$$

Since a and b are perpendicular $\vec{a}.\vec{b}$ = 0

(2) (1) + (λ) (-2) + (1) (1) = 0

$2 - 2\lambda + 1 = 0 = \lambda = \frac{3}{2}$

4. Find the value of m for which the vectors $\vec{a} = 3\vec{i} + 2\vec{j} + 9\vec{k}$ and $\vec{b} = \vec{i} + m\vec{j} + 3\vec{k}$ are (i) perpendicular, 9ii) parallel.

Solution: $\vec{a} = 3\vec{i} + 2\vec{j} + 9\vec{k}$

$\vec{b} = \vec{i} + m\vec{j} + 3\vec{k}$

(i) If they are perpendicular $\vec{a}.\vec{b} = 0$

Hence (3) (1) + (2) (m) + (9) (3) = 0

3 + 2m + 27 = 0

= m = - 15

(ii) If they are parallel, $\frac{3}{1} = \frac{2}{m} = \frac{9}{3}$

= 9m = 6 => m = $\frac{2}{3}$

5. Find the angles which the vector $\vec{i} - \vec{j} + \sqrt{2}\ \vec{k}$ makes with the coordinate axes.

Solution: Let $\vec{F} = \vec{i} - \vec{j} + \sqrt{2}\ \vec{k}$

$| F | = \sqrt{(1)^2(-1)^2 + (\sqrt{1})^2} = 2$

Hence direction cosines *l*, *m*, *n* of F are

$l = \frac{a}{|F|} = \frac{1}{2}$, m = $\frac{b}{|F|} = \frac{1}{2}$, n = $\frac{c}{|F|} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

Let α, β and γ are the angles at which r makes with x-axis, y-axis and z-axis, then

$\cos \alpha = l = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}$

$\cos \beta = m = \frac{1}{2} \Rightarrow \beta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\cos \gamma = n = \frac{1}{\sqrt{2}} \Rightarrow \gamma = \frac{\pi}{4}$

6. Show that the vector $\vec{i} + \vec{j} + \vec{k}$ is equally inclined with the coordinate axes.

Solution: $\vec{F} = \vec{i} + \vec{j} + \vec{k}$

$|F| = \sqrt{(1)2 + (1)2 + (1)2} = \sqrt{3}$

Hence the direction cosines *l*, *m*, *n* of F are

$l = \frac{a}{|F|} = \frac{1}{\sqrt{3}}$, m = $\frac{b}{|F|} = \frac{1}{\sqrt{3}}$, n = $\frac{c}{|F|} = \frac{1}{\sqrt{3}}$

Let α, β and γ be the angles at which $\bar{r}$ is inclined to *x*-axis and *z*-axis.

Then, $\cos \alpha = \frac{1}{\sqrt{3}}$, $\cos \beta = \frac{1}{\sqrt{3}}$, $\cos \gamma = \frac{1}{\sqrt{3}}$

` $\alpha = \beta = \gamma = \cos^{-1} \frac{1}{\sqrt{3}}$ $\left(\quad \right)$

7. If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ, then prove that

(i) $\cos\frac{\theta}{2} = \frac{1}{2}\ |\hat{a}\ +\hat{b}\ |$ and (ii) $\tan\ \frac{\theta}{2}\ =\ \frac{|\hat{a}-\hat{b}\,|}{|\hat{a}+\hat{b}|}$

Solution: (i) $|\hat{a}\ +\hat{b}\ |^2\ = |a|^2 + |b|^2 + 2|a, b|$

$= 1 + 1 + 2\ |a|\ |b|\ \cos\theta$

$= 2 + 2\ (1)\ (1)\cos\theta \qquad = 2 + 2\cos\ \theta$

$= 2\ (1 + \cos\theta) = 2\ \left(2\cos^2\frac{\theta}{2}\right)$

$|\ \hat{a}\ +\hat{b}\ \ |^2 = 4\cos^2\frac{\theta}{2}$

$\frac{1}{4}\ |\ \hat{a}\ +\hat{b}\ \ |^2 = \cos^2\frac{\theta}{2}$

$\frac{1}{2}\ \ |\ \hat{a}\ +\hat{b}\ \ |\ = \cos\ \frac{\theta}{2}$

(ii) From the above result, we get $|\hat{a}\ +\hat{b}\ | = 2\cos\frac{\theta}{2}$,

$|\hat{a}\ -\ \hat{b}\ | = 2\sin\frac{\theta}{2}$ then $\tan\ \frac{\theta}{2}\ =\ \frac{|\hat{a}-\hat{b}\,|}{|\hat{a}+\hat{b}|}$.

8. If the sum of two unit vectors is a unit vector prove that the magnitude of their difference is $\sqrt{3}$.

Solution: Let $\hat{a}\ +\hat{b} = \hat{c}$ given $|\hat{c}| = 1$, also a, b are unit vectors.

To prove that: $|a - b|\ = \sqrt{3}$

$(\vec{a}\ +\ \vec{b}\ ) . (\vec{a}\ +\ \vec{b})\ =\ \vec{a}.\vec{a} + 2\vec{a}.\vec{b}\ +\ \vec{b}.\vec{b}$

$$|c|^2 = |a|^2 + 2a.b + |b|^2$$

$$\Rightarrow 1 = |a|^2 + 2(a.b) + |b|^2$$

$$\Rightarrow |a|^2 + |b|^2 = 1 - 2(a.b)$$

$$\Rightarrow 2 = 1 - 2(a.b)$$

$$1 = -2(a.b)$$

Now, $(\vec{a} - \vec{b}) = a.a - 2(a.b) + b.b$

$$|\vec{a} - \vec{b}|^2 = |a|^2 + |b|^2 - 2(a.b)$$

$$= 1 + 1 + 1$$

$$= 3$$

$$\Rightarrow |\vec{a} - \vec{b}| = \sqrt{3}.$$

9. If a, b, c are three mutually perpendicular unit vectors, then prove that

$$|\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}.$$

Solution: Given $\vec{a}$, $\vec{b}$, $\vec{c}$ are three mutually perpendicular unit vectors.

$\Rightarrow |\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

and $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$

Now, $(\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{c} = 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$ since $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$

$|\vec{a} + \vec{b} + \vec{c}|^2 = 1 + 1 + 1 = 3$

$\Rightarrow |\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}.$

10. If $|\vec{a} + \vec{b}| = 60$, $|\vec{a} - \vec{b}| = 40$ and $|\vec{b}| = 46$ find $|\vec{a}|$.

Solution: $|\vec{a} + \vec{b}| = 60$, $|\vec{a} - \vec{b}| = 40$, $|\vec{b}| = 46$

$3600 + 1600 = 2|a|^2 + 4232$

2| a |2 = 968 = | a |2 = 484

` | a | = 22.

11. Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vector such that $\vec{u} + \vec{v} + \vec{w} = 0$.

If $|\vec{u}| = 3$, $|\vec{v}| = 4$ and $|\vec{w}| = 5$ then find $\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{u}$

Solution:

$(\vec{u}+\vec{v}+\vec{w}).(\vec{u}+\vec{v}+\vec{w}) = |\vec{u}|^2 + |\vec{v}|^2 + |\vec{w}|^2 + 2(\vec{u}.\vec{v}+\vec{u}.\vec{w}+\vec{w}.\vec{u})$

$\Rightarrow 0 = 9 + 16 + 25 + 2(\vec{u}.\vec{v}+\vec{u}.\vec{w}+\vec{w}.\vec{u})$

$\Rightarrow 2(\vec{u}.\vec{v}+\vec{u}.\vec{w}+\vec{w}.\vec{u}) = -50$

$\vec{U}.\vec{v}+\vec{u}.\vec{w}+\vec{w}.\vec{u} = -25.$

12. Show that the vectors $3\vec{i} - 2\vec{j} + \vec{k}$, $\vec{i} - 3\vec{j} + 5\vec{k}$ and $2\vec{i} + \vec{j} - 4\vec{k}$ form a right angled triangle.

Solution: Let $\vec{a} = 3\vec{i} - 2\vec{j} + \vec{k}$

$\vec{b} = \vec{i} - 3\vec{j} + 5\vec{k}$

and $\vec{c} = 2\vec{i} + \vec{j} - 4\vec{k}$

$\vec{a}.\vec{b} = (3)(1) + (-2)(-3) + (1)(5) = 3 + 6 + 5 = 14$

$b.c = (1)(2) + (-3)(1) + (5)(-4) = 2 - 3 - 20 = -21$

$c.a = (3)(2) + (-2)(1) + (1)(-4) = 6 - 2 - 4 = 0$

$\Rightarrow$ c and a are perpendicular to each other.

Also, $\vec{b} + \vec{c} = (\vec{i} - 3\vec{j} + 5\vec{k}) + (2\vec{i} + \vec{j} - 4\vec{k})$

$= 3\vec{i} - 2\vec{j} + \vec{k} = \vec{a}$

Hence the vectors form a right angled triangle.

Another method:

$|\vec{a}| = \sqrt{(3)^2 + (-2)^2 + (1)^2} = \sqrt{9+4+1} = \sqrt{14}$

$|\vec{b}| = \sqrt{(1)^2 + (-3)^2 + (5)^2} = \sqrt{1 + 9 + 25} = \sqrt{35}$

$|\vec{c}| = \sqrt{(2)^2 + (1)^2 + (-4)^2} = \sqrt{4 + 1 + 16} = \sqrt{21}$

Since $|\vec{b}|^2 = |\vec{a}|^2 + |\vec{c}|^2$

The vectors form a right angled triangle.

13Show that the points whose position vectors $4\vec{i} - 3\vec{j} + \vec{k}$, $2\vec{i} - 4\vec{j} + 5\vec{k}$, $\vec{i} - \vec{j}$ Form a right angled triangle.

Solution: Let $\overline{OA} = 4\vec{i} - 3\vec{j} + \vec{k}$

$\overline{OB} = 2\vec{i} - 4\vec{j} + 5\vec{k}$ $\quad \overline{OC} = \vec{i} - \vec{j}$

$\overline{AB} = \overline{OB} - \overline{OA}$

$= 2\vec{i} - 4\vec{j} + 5\vec{k} - (4\vec{i} - 3\vec{j} + \vec{k}) = 2\vec{i} - \vec{j} + 4\vec{k}$

$\overline{BC} = \overline{OC} - \overline{OB}$

$= (\vec{i} - \vec{j}) - (2\vec{i} - 4\vec{j} + 5\vec{k}) - \vec{i} + 3\vec{j} - 5\vec{k}$

$\overline{CA} = \overline{OA} - \overline{OC}$

$= (4\vec{i} - 3\vec{j} + \vec{k}) - (\vec{i} - \vec{j}) = 3\vec{i} - 2\vec{j} + \vec{k}$

$|\overline{AB}| = \sqrt{(-2)^2 + (-1)^2 + (4)^2} = \sqrt{4 + 1 + 16} = \sqrt{21}$

$|\overline{BC}| = \sqrt{(-1)^2 + (3)^2 + (-5)^2} = \sqrt{1 + 9 + 25} = \sqrt{35}$

$|\overline{CA}| = \sqrt{(3)^2 + (-2)^2 + (1)^2} = \sqrt{9 + 4 + 1} = \sqrt{41}$

$|\overline{BC}|^2 = |\overline{AB}|^2 + |\overline{CA}|^2$

$\Rightarrow 35 = 21 + 14 \Rightarrow 35 = 35$

⇒ The triangle is right angled. $\overline{AB} + \overline{BC} = \overline{AC}$.

14. Find the projection of

(i) $\vec{i} - \vec{j}$ on z-axis, (ii) $\vec{i} + 2\vec{j} - 2\vec{k}$ on $2\vec{i} - \vec{j} + 5\vec{k}$, (iii) $3\vec{i} + \vec{j} - \vec{k}$ on $4\vec{i} - \vec{j} + 2\vec{k}$.

Solution: (i) Projection of $\vec{i} - \vec{j}$ on z-axis = $\frac{(\vec{i}-\vec{j})\ \ \vec{k}}{|\vec{k}|}$ 0

(ii) Projection of $\vec{i} + 2\vec{j} - 2\vec{k}$ on $2\vec{i} - \vec{j} + 5\vec{k}$ is $\frac{(\vec{i}+2\vec{j}-2\vec{k})(2\vec{i}-\vec{j}+5\vec{k})}{|2\vec{i}-\vec{j}+5\vec{k}|}$

$= \frac{2-2-10}{\sqrt{4+1+25}} = \frac{-10}{\sqrt{30}}$

(iii) Projection of $3\vec{i} + \vec{j} - \vec{k}$ on $4\vec{i} - \vec{j} + 2\vec{k}$ is $\frac{(3\vec{i}+\vec{j}-\vec{k}).(4\vec{i}-\vec{j}+2\vec{k})}{|4\vec{i}-\vec{j}+2\vec{k}|}$

$= \frac{12-1-2}{\sqrt{16+1+4}} = \frac{9}{\sqrt{21}}$

EXERCISE 2.2

Prove by vector method.

1. If the diagonals of a parallelogram are equal then it is a rectangle.

Solution: Let ABCD be a parallelogram. Let AC and BD be the diagonals

Then AC = $\vec{BD}$ (given)

$\Rightarrow |\vec{AC}|^2 = |\vec{BD}|^2$

$\Rightarrow \vec{AC} . \vec{AC} = \vec{BD} . \vec{BD}$

$(\vec{AB} + \vec{BC}). (\vec{AB} + \vec{BC}) = (\vec{BC} + \vec{CD}) . (\vec{BC} + \vec{CD})$

$= (\vec{BC} - \vec{AB}) . (\vec{BC} - \vec{AB})$

$\Rightarrow |\vec{AB}|^2 + |\vec{BC}|^2 + 2\vec{AB} . \vec{BC} = |\vec{BC}|^2 + |\vec{AB}|^2 - 2\vec{BC} . \vec{AB}$

$=.> 4 \vec{AB} . \vec{BC} = 0$

Hence $\vec{AB}$ is perpendicular to $\vec{BC}$

=> ABCD us a rectangle.

2. The mid point of the hypotenuse of a right angled triangle is equidistant from its vertices.

Solution: Given ABC is a right angled triangle in which AC is the hypotenuse and D is the mid point of AC.

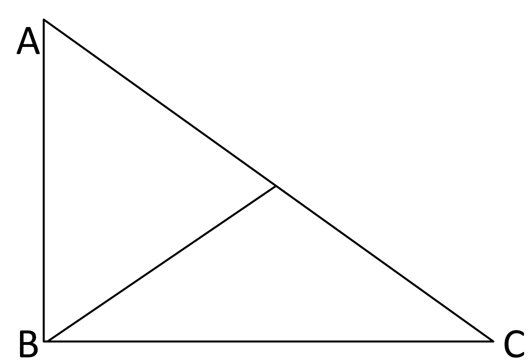
$\Rightarrow \quad \vec{AD} = \vec{DC}$

Since B = 90

$\vec{AB}, \vec{BC} = 0$

But $\vec{AB} = \vec{AD} + \vec{DB}$

And $\vec{BC} = \vec{BD} + \vec{DC} = - \vec{DB} + \vec{AD}$



Hence from (i). $(\vec{AD} + \vec{DB}). (- \vec{DB} + \vec{AD}) = 0$

$\Rightarrow |\vec{AD}|^2 - |\vec{DB}|^2 = 0$

$\Rightarrow |\vec{AD}| = |\vec{DB}|$

Hence $|\vec{AD}| = |DC| = |DB|$

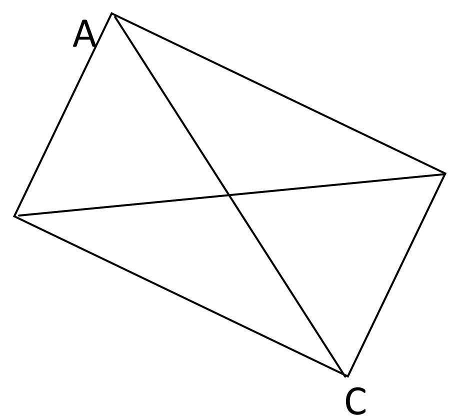
D is equidistant from the vertices.

3. The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the sides.

Solution: $\vec{AC} = \vec{AB} + \vec{BC}$

$\vec{BD} = \vec{BA} + \vec{AD}$

$= \vec{AD} + \vec{BA} = \vec{AD} - \vec{AB}$



$\vec{AC}^2 = (\vec{AB} + \vec{BC})^2 D$

$= \vec{AB}^2 + \vec{BC}^2 + 2\,\vec{AB}.\,\vec{BC}$

$= \vec{AB}^2 + \vec{BC}^2 + 2\vec{AB} + \vec{AD}$

$\vec{BD}^2 = (\vec{AD} - \vec{AB})^2$

$= \vec{AD}^2 + \vec{AB}^2 - 2\vec{AB}.\,\vec{AD}$

$\vec{AC}^2 + \vec{BD}^2 = \vec{AB}^2 + \vec{BC}^2 + \vec{AD}^2 + \vec{AB}^2$

$= \vec{AB}^2 + \vec{BC}^2 + \vec{DC}^2 + \vec{AD}^2$

4. cos (A+B) = cos A cos B B – sin A sin B.

Solution: Let $\vec{I}$, $\vec{j}$ be the unit vectors along OX and OY, OP and OQ are draw such that XOP = A and XOQ = B so that POQ = $\vec{a} + \vec{b}$

Take $\vec{OM} = \vec{OL}$ = 1 unit

Draw MN ⊥ to OX

$\rightarrow \quad \rightarrow \quad \rightarrow$

$OM = ON + NM$

$\vec{OM} = \text{COS } A\ \vec{i} + \sin A\ \vec{j}$

$\vec{OL} = \cos B\ \vec{i} - \sin B\ \vec{j}$

$\vec{OM} . \vec{OL} = (\cos A\ \vec{i} + \sin A\ \vec{j}). (\cos B\ \vec{i} - \sin B\ \vec{j})$

$|\vec{OM}|\ |\vec{OL}| \cos (A + B) = \cos A \cos B - \sin A \sin B$

$=> \text{Cos } (A + B) = \cos A \cos B - \sin A \sin B$

5. Find the work done by the force $\vec{F} = 2\vec{i} + \vec{j} + \vec{k}$ acting on a particle, if the particle is displaced from the point with position vector $2\vec{i} + 2\vec{j} + 2\vec{k}$ to the point with Position vector $3\vec{i} + 4\vec{j} + 5\vec{k}$.

Solution: Displacement $\vec{d} = \vec{AB} = \vec{OB} - \vec{OA}$

$(\vec{OA} = 2\vec{i} + \vec{j} + \vec{k};\ \vec{OB} = 3\vec{i} + 4\vec{j} + 5\vec{k})$

$= (3\vec{i} + 4\vec{j} + 5\vec{k}) - (2\vec{i} + 2\vec{j} + 2\vec{k})$

$= (\vec{i} + 2\vec{j} + 3\vec{k})$

Work done $= \vec{F} . \vec{d}$

$= (2\vec{i} + \vec{j} + \vec{k}). (\vec{i} + 2\vec{j} + 3\vec{k})$

$= 2 + 2 + 3 = 7$ units.

6. A force of magnitude 5 units acting parallel of $2\vec{i} - 2\vec{j} + \vec{k}$ displaces the point of application from (1,2,3) to 5,3,7). Find the work done.

Solution: Displacement $= \vec{AB} = \vec{OB} - \vec{OA}$

$= (\vec{OA} = \vec{i} + 2\vec{j} + 3\vec{k};\ \vec{OB} = 5\vec{i} + 3\vec{j} + 7\vec{k})$

$$= \quad 4\vec{i} + \vec{j} + 4\vec{k}$$

Force of magnitude 5 units acting parallel to $2\vec{i} - 2\vec{j} + \vec{k}$

$$= \quad 5 \frac{2\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{4+4+1}} = \frac{5}{3}(2\vec{i} - 2\vec{j} + \vec{k})$$

$$= \frac{10}{3}(4) - \frac{10}{3}(1) + \frac{5}{3}(4)$$

$$= \frac{40}{3} - \frac{10}{3} + \frac{20}{3} = \frac{50}{3}$$

7. The constant forces $2\vec{i} - 5\vec{j} + 6\vec{k}$, $-\vec{i} + 2\vec{j} - \vec{k}$ and $2\vec{i} + 7\vec{j}$ act on a particle which is displaced from position $4\vec{i} - 3\vec{j} - 2\vec{k}$ to position $6\vec{i} + \vec{j} - 3\vec{k}$. Find the work done.

Solution: Displacement = Final position – Initial position

$$= (6\vec{i} + \vec{j} - 3\vec{k}) - (4\vec{i} - 3\vec{j} - 2\vec{k})$$

$$= 2\vec{i} + 4\vec{j} - \vec{k}$$

Total forces $= (2\vec{i} - 5\vec{j} + 6\vec{k}) + (-\vec{i} + 2\vec{j} - \vec{k}) + (2\vec{i} + 7\vec{j})$

$$= \quad (3\vec{i} + 4\vec{j} + 5\vec{k})$$

Work done $= \vec{F}.\vec{d}$

$$= (3\vec{i} + 4\vec{j} + 5\vec{k}).(2\vec{i} + 4\vec{j} - \vec{k})$$

$$= 6 + 16 - 5 = 17$$

8. Forces of magnitudes 3 and 4 units acting in directions $6\vec{i} + 2\vec{j} + 3\vec{k}$ and $3\vec{i} - 2\vec{j} + 6\vec{k}$ respectively act on a particle which is displaced from the point (2, 2, - 1) to (4, 3, 10. Find the work done by the forces.

Solution: Displacement = Final position - Initial positions

$$= (4\vec{i} + 3\vec{j} + \vec{k}) - (2\vec{i} + 2\vec{j} - \vec{k})$$

$$= 2\vec{i} + \vec{j} + 2\vec{k}$$

$$\text{Forces are} = 3\left(\frac{6\vec{i}+2\vec{j}+3\vec{k}}{\sqrt{36+4+9}}\right) \text{ and } 4\left(\frac{3\vec{i}+2\vec{j}+6\vec{k}}{\sqrt{9+4+36}}\right)$$

$$= \frac{3}{7}(6\vec{i} + 2\vec{j} + 3\vec{k}) \text{ and } \frac{4}{7}(3\vec{i} - 2\vec{j} + 6\vec{k})$$

$$\text{Sum of the forces} = \frac{3}{7}(6\vec{i} + 2\vec{j} + 3\vec{k}) + \frac{4}{7}(3\vec{i} - 2\vec{j} + 6\vec{k})$$

$$= \frac{1}{7}(18\vec{i} + 6\vec{j} + 9\vec{k}) + \frac{1}{7}(12\vec{i} - 8\vec{j} + 24\vec{k})$$

$$= \frac{1}{7}(30\vec{i} - 2\vec{j} + 33\vec{k})$$

$$\therefore \text{ work done} = \vec{F}.\vec{d}$$

$$= \frac{1}{7}(30\vec{i} - 2\vec{j} + 33\vec{k}).(2\vec{i} + \vec{j} + 2\vec{k})$$

$$= \frac{1}{7}[30(2) - 2(1) + 33(2)]$$

$$= \frac{1}{7}[60 - 2 + 66] = \frac{124}{7} = \frac{124}{7} \text{ units.}$$

SOLUTIONS OF EXERCISE – 2.3

1. Find the magnitude of $\vec{a} \times \vec{b}$ if $\vec{a} = 2\vec{i} + \vec{k}$, $\vec{b} = \vec{i} + \vec{j} + \vec{k}$

Solution: Let $\vec{a} = 2\vec{i} + \vec{k}$; $\vec{b} = \vec{i} + \vec{j} + \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \vec{i}\,(0-1) - \vec{j}\,(2-1) + \vec{k}\,(2-0)$$

$$= -\vec{i} - \vec{j} + 2\vec{k}$$

$$\therefore\ |a \times b| = \sqrt{(-1)^2 + (-1)^2 + (2)^2} = \sqrt{1+1+4} = \sqrt{6}$$

2. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $\vec{a}.\vec{b} = 9$ then find $|\vec{a} \times \vec{b}|$

Solution: $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

$$\therefore\ 9 = 3 \times 4 \cos\theta$$

Hence $\sin\theta = \sqrt{1 - cos^2\theta}$

$$= \sqrt{1 \frac{9}{16}} = \frac{\sqrt{7}}{4}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$$

$$|\vec{a} \times \vec{b}| = 3 \times 4 \times \frac{\sqrt{7}}{4} = 3\sqrt{7}$$

3. Find the unit vectors perpendicular to the plane containing the vectors $2\vec{i} + \vec{j} + \vec{k}$ and $\vec{i} + 2\vec{j} + \vec{k}$.

Solution: $\vec{a} = 2\vec{i} + \vec{j} + \vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} + \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$= \vec{i}(1-2) - \vec{j}(2-1) + \vec{k}(4-1)$

$= -\vec{i} - \vec{j} + 2\vec{k}$

$$\therefore \vec{n} = \pm\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \pm\frac{-\vec{i}-\vec{j}+3\vec{k}}{\sqrt{1+1+9}} = \pm\frac{-\vec{i}-\vec{j}+3\vec{k}}{\sqrt{11}}$$

4. Find the vectors whose length 5 and which are perpendicular to the vectors

$\vec{a} = 3\vec{i} + \vec{j} - 4\vec{k}$ and $\vec{b} = 6\vec{i} + 5\vec{j} - 2\vec{k}$.

Solution: $\vec{a} = 3\vec{i} + \vec{j} - 4\vec{k}$

$\vec{b} = 6\vec{i} + 5\vec{j} - 2\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -4 \\ 6 & 5 & -2 \end{vmatrix}$$

$= \vec{i}(-2 + 20) - \vec{j}(-6 + 24) + \vec{k}(15 - 6)$

$= 18\vec{i} - 18\vec{j} + 9\vec{k}$

$$|\vec{a} \times \vec{b}| = \sqrt{(18)^2 + (-18)^2 + (9)^2}$$

$$= \sqrt{324 + 324 + 81} = \sqrt{729}$$

∴ Vectors whose length 5 and which are perpendicular to a and b is

$$\vec{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

$$= \frac{5(18\vec{i} - 18\vec{j} + 9\vec{k})}{\sqrt{729}} = \frac{90\vec{i} - 90\vec{j} + 45\vec{k}}{\sqrt{27}}$$

$$= \frac{10\vec{i} - 10\vec{j} + 5\vec{k}}{3} = \frac{10\vec{i} - 10\vec{j} + 5\vec{k}}{3}$$

5. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ if $|\vec{a} \times \vec{b}| = \vec{a} . \vec{b}.$

Solution: $|a \times b| = a . b$

$|a|\,|b| \sin\theta = |a|\,|b| \cos\theta$

$\Rightarrow \frac{\sin\theta}{\cos\theta} \Rightarrow 1$

$\Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

6. If $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $\vec{a} \times \vec{b} = 3\vec{i} - 2\vec{j} + 6\vec{k}$ find angle between $\vec{a}$ and $\vec{b}$

Solution: $\vec{a} \times \vec{b} = 3\vec{i} - 2\vec{j} + 6\vec{k}$

$\therefore |\vec{a} \times \vec{b}| = \sqrt{9 + 4 + 36} = \sqrt{49} = 7$

$|\vec{a}|\,|\vec{b}| \sin\theta = 7$

$$2 \times 7 \times \sin\theta = 7$$

$$\Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

7. If $\vec{a} = \vec{i} + 3\vec{j} - 2\vec{k}$ and $\vec{b} = -\vec{i} + 3\vec{k}$ then find $\vec{a} \times \vec{b}$. Verify that $\vec{a}$ and $\vec{b}$

Solution: $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix}$

$$= \vec{i}(9-0) - \vec{j}(3-2) + \vec{k}(0+3)$$

$$= 9\vec{i} - \vec{j} + 3\vec{k}$$

$$\vec{a}\cdot(\vec{a} \times \vec{b}) = (\vec{i} + 3\vec{j} - 2\vec{k})\cdot(9\vec{i} - \vec{j} + 3\vec{k})$$

$$= 9 - 3 - 6 = 0$$

$\Rightarrow \vec{a}$ and $(\vec{a} \times \vec{b})$ are perpendicular

$$\vec{b}\cdot(\vec{a} \times \vec{b}) = (-\vec{i} + 3\vec{k})\cdot(9\vec{i} - \vec{j} + 3\vec{k})$$

$$= -9 + 0 + 9 = 0$$

$\Rightarrow \vec{b}$ and $(\vec{a} \times \vec{b})$ are perpendicular

8. For any three vectors $\vec{a}, \vec{b}, \vec{c}$ show that

$\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = 0.$

Solution: $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$

$$= (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c}) + (\vec{b} \times \vec{a})$$

$$+ (\vec{c} \times \vec{a}) + (\vec{c} \times \vec{b})$$

$= \vec{0}$

Since $(\vec{a} \times \vec{b}) = -(\vec{b} \times \vec{a})$

$(\vec{a} \times \vec{c}) = -(\vec{c} \times \vec{a})$

and $(\vec{b} \times \vec{c}) = -(\vec{c} \times \vec{b})$

9. Let $\vec{a}, \vec{b}, \vec{c}$ be unit vectors such that $\vec{a}.\vec{c} = 0$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$. Prove that $\vec{a} = \pm 2(\vec{b} \times \vec{c})$.

Solution: Given : $\vec{a}.\vec{b} = 0$ and $\vec{a}.\vec{c} = 0$

Angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$

$=> \vec{a}$ is perpendicular to the plane containing $\vec{b}$ and $\vec{c}$ and the angle between $\vec{b}$ and $\vec{c}$ is (in other words $\vec{n} = \vec{a}$)

$\therefore \vec{b} \times \vec{c} = |\vec{b}|\,|\vec{c}| \sin\theta$ n where θ is the angle between $\vec{b}$ and $\vec{c}$

$= 1 \times 1 \sin \frac{\pi}{6}\ \vec{a}$ since $\vec{b}, \vec{c}$ are unit vectors

$= \frac{1}{2}\ \vec{a} => 2(\vec{b} \times \vec{c})$ or in general $\vec{a}\ \pm\ 2(\vec{b} \times \vec{c})$

10. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$

Show that $\vec{a} - \vec{d}$ and $\vec{b} - \vec{c}$ are parallel.

Solution

$(\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c}) - (\vec{d} \times \vec{b}) + (\vec{d} \times \vec{c})$

$= (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{d}) - (\vec{c} \times \vec{d})$

$= 0$

=> $(\vec{a} - \vec{d})$ and $(\vec{b} - \vec{c})$ are parallel.

EXERCISE 2.4

1. Find the area of parallelogram ABCD whose vertices are

A (-5,2,5), B(-3,6,7), C(4,-1,5) and D(2,-5,3)

Solution:

Let O be the point of reference and $\vec{OA} = -5\vec{i} + 2\vec{j} + 5\vec{k}$.

$\vec{OB} = -3\vec{i} + 6\vec{j} + 7\vec{k}$ $\quad \vec{OC} = 4\vec{i} - \vec{j} + 5\vec{k}$ and $\quad \vec{OD} = 2\vec{i} - 5\vec{j} + 3\vec{k}$

Area of parallelogram ABCD = $|\vec{AB} \times \vec{AC}|$

$\vec{AB} = \vec{OB} - \vec{OA} = 2\vec{i} + 4\vec{j} + 2\vec{k}$

$\vec{AC} = \vec{OC} - \vec{OA} = 9\vec{i} - 3\vec{j}$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 4 & 2 \\ 9 & -3 & 0 \end{vmatrix} = 6\vec{i} + 18\vec{j} - 42\vec{k}$$

$= 6(\vec{i} + 3\vec{j} - 7\vec{k})$

$|\vec{AB} \times \vec{AC}| = 6\sqrt{59}$.

2. Find the area of the parallelogram whose diagonals are represented by

$2\vec{i} + 3\vec{j} + 6\vec{k}$ and $3\vec{i} - 6\vec{j} + 2\vec{k}$

Solution:

Let $\vec{d_1} = 2\vec{i} + 3\vec{j}\ 6\vec{k}$ $\quad \vec{d_2} = 3\vec{i} - 6\vec{j} + 2\vec{k}$

Area of parallelogram $= \frac{1}{2} |\vec{d_1} \times \vec{d_2}|$

$$\vec{d_1} \times \vec{d_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 6 \\ 3 & -6 & 2 \end{vmatrix} = 42\vec{i} + 14\vec{j} + 21\vec{k}$$

$$= 7(6\vec{i} + 2\vec{j} - 3\vec{k}) = 7 \times |6\vec{i} + 2\vec{j} - 3\vec{k}|$$

$$\frac{1}{2} |d_1 \times d_2| = \frac{7}{2}\sqrt{(6)^2 + (2)^2 + (-3)^2}$$

$$= \frac{7}{2}\sqrt{49} = \frac{49}{2} \text{ sq. units.}$$

3. Find the area of the parallelogram determined by the sides

$\vec{i} + 2\vec{j} + 3\vec{k}$ and $-3\vec{i} - 2\vec{j} + \vec{k}$

Solution:

Let $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{b} = 3\vec{i} - 2\vec{j} + \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{vmatrix} = 8\vec{i} - 10\vec{j} + 4\vec{k}$$

$$\text{Area} = \vec{a} \times \vec{b} = \sqrt{(8)^2 + (-10)^2 + (-4)^2}$$

$$= \sqrt{180} = 6\sqrt{5} \text{ sq. units.}$$

4. Find the area of the triangle whose vertices are (3, -1, 2), (1, -1, -3) and

(4, -3, 1)

Solution:

Let ABC be the given triangle and let $\vec{OA} = 3\vec{i} - \vec{j} + 2\vec{k}$

$\vec{AB} = \vec{OB} - \vec{OA} = 2\vec{i} - 5\vec{k}$

$\vec{AC} = \vec{OC} - \vec{OA} = \vec{i} - 2\vec{j} - \vec{k}$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix} = -10\vec{i} - 7\vec{j} + 4\vec{k}$$

$$\frac{1}{2} | \vec{AB} \times \vec{AC} | = \frac{1}{2} | -10\vec{i} - 7\vec{j} + 4\vec{k} |$$

$$= \frac{1}{2}\sqrt{(-10)^2 + (-7)^2 + (4)^2}$$

$$= \frac{1}{2}\sqrt{165} \text{sq. units.}$$

5. Prove by vector method that the parallelograms on the same base and between the same parallels are equal in area.

Solution:

Let ABCD be the given parallelogram and

ABCD be the new parallelogram with same

Base AB and between the same parallel lines $\vec{AB}$ and $\vec{DC}$

The vector area of ABCD $= \vec{AB} \times \vec{AD}$

$= \vec{AB} \times (\vec{AD'} + \vec{DD'})$

$= (\vec{AB} \times A'D) + \vec{AB} + 0$

= vector area of ABCD

i. e. area of ABCD = area of ABCD'

6. Prove that twice the area of a parallelogram is equal to the area of another parallelogram formed by taking as its adjacent sides the diagonals of the former parallelogram.

Solution: Let ABCD be the given parallelogram

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BC} = \overrightarrow{BC} - \overrightarrow{AB}$

Area of the parallelogram with AC and BD as adjacent sides

$= |\overrightarrow{AC} \times \overrightarrow{BD}|$

$= |(\overrightarrow{AB} + \overrightarrow{BC}) \times (\overrightarrow{BC} - \overrightarrow{AB})|$

$= |\overrightarrow{AB} \times \overrightarrow{BC} - \overrightarrow{AB} \times \overrightarrow{AB} + \overrightarrow{BC} \times \overrightarrow{BC} - \overrightarrow{BC} \times \overrightarrow{AB}|$

$= |\overrightarrow{AB} \times \overrightarrow{BC} + \overrightarrow{AB} \times \overrightarrow{BC}| = 2\,|\overrightarrow{AB} \times \overrightarrow{BC}|$

= 2 (area of the parallelogram ABCD)

7. Prove that sin (A – B) = sin A cos B – cos A sin B.

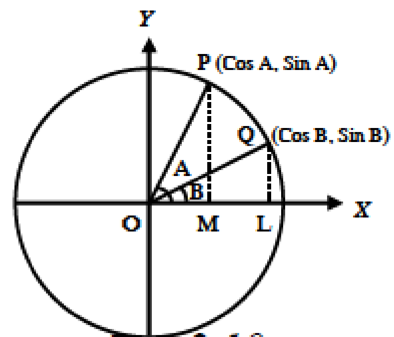


Fig. 2.10

Solution:

Take the points P and Q on the unit circle with centre at the origin O. Assume that OP and OQ make angles. A and B with x-axis respectively.

$\angle POQ = \angle POx + \angle QOx = A - B$

Clearly the co-ordinates of P and Q are (sos A. sin A) and (cos B, sin B).

Take the unit vectors $\vec{i}$ and $\vec{j}$ along x and axes respectively.

$$\vec{OP} = \vec{OM} + \vec{MP}$$

$$= \cos A\,\vec{i} + \sin A\,\vec{j}$$

$$\vec{OQ} = \vec{OL} + \vec{LQ}$$

$$= \cos B\vec{i} + \sin B\,\vec{j}$$

$$\vec{OQ} \times \vec{OP} = |\vec{OQ}|\,|\vec{OP}| \sin(A - B)\,\vec{k} = \sin(A - B)\,\vec{k}$$

$$OQ \times OP = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos B & \sin B & 0 \\ \cos A & \sin A & 0 \end{vmatrix} = \vec{k}\,[\sin a \cos B - \cos A \sin B]$$

From (1) and (2)

$$\sin (A - B) = \sin A \cos B - \cos A \sin B$$

8. Forces $2\vec{i} + 7\vec{j}$, $2\vec{i} - 5\vec{j} + 6\vec{k}$, $\vec{i} + 2\vec{j} - \vec{k}$ act at a point P whose position vector is $4\vec{i} - 3\vec{j} - 2\vec{k}$. find the moment of the resultant of three forces acting at P about the point Q whose position vector $6\vec{i} + \vec{j} - 3\vec{k}$.

Solution: The resultant force $\vec{F} = \vec{F1} + \vec{F2} + \vec{F3}$

$$\vec{F} = (2\vec{i} + 7\vec{j}) + (2\vec{i} - 5\vec{j} + 6\vec{k}) + (-\vec{i} + 2\vec{j} - \vec{k})$$

$$= 3\vec{i} + 4\vec{j} + 5\vec{k}$$

Let $\vec{OP} = 4\vec{i} - 3\vec{j} - 2\vec{k}$ and $\vec{OQ} = 6\vec{i} + \vec{j} - 3\vec{k}$

$\vec{r} = \vec{OP} - \vec{OQ}$ [through (or at) – about]

$$= -2\vec{i} - 4\vec{j} + \vec{k}$$

Moment $\vec{M} = \vec{r} \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -4 & 1 \\ 3 & 4 & 5 \end{vmatrix}$$

$$\vec{M} = -24\vec{i} + 13\vec{j} + 4k$$

9. Show that torque about the point A(3, -1, 3) of a force $4\vec{i} + 2\vec{j} + \vec{k}$ through the point B (5, 2, 4) is $\vec{i} + 2\vec{j} - 8\vec{k}$.

Solution:

Let $\vec{F} = 4\vec{i} + 2\vec{j} + k$

Let $\vec{OA} = 3\vec{i} - \vec{j} + 3\vec{k}$ and $OB = 5\vec{i} + 2\vec{j} + 4\vec{k}$

$\vec{r} = \vec{OB} - \vec{OA} = 2\vec{i} + 3\vec{j} + \vec{k}$

Torque (moment) $\vec{M} = \vec{r} \times \vec{F}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ 4 & 2 & 1 \end{vmatrix}$$

Torque $= \vec{i} + 2\vec{j} - 8\vec{k}$

10. Find the magnitude and direction cosines of the moment about the point

(1, - 2, 3) of a force $2\vec{i} + 3\vec{j} + 6\vec{k}$ whose line of action passes through the

origin.

Solution: $\vec{F} = 2\vec{i} + 3\vec{j} + 6\vec{k}$

Let $\vec{OP} = \vec{O}$ AND $\vec{OA} = \vec{i} - 2\vec{j} + 3\vec{k}$

$R = \vec{OP} - \vec{OA} = -\vec{i} + 2\vec{j} - 3\vec{k}$

$\vec{M} = \vec{r} \times \vec{F}$

$$= \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & -3 \\ 2 & 3 & 6 \end{matrix} = 21\vec{i} - 7\vec{k}$$

$$|\vec{r} \times \vec{F}| = \sqrt{(21)^2 + (-7)^2} = 7\sqrt{10}$$

d.c.s of the moment are $\left\{\frac{21}{\sqrt{10}}, \ 0, \ \frac{-7}{7\sqrt{10}}\right\}$ i.e., $\left\{\frac{3}{\sqrt{10}}, \ 0, \ \frac{1}{\sqrt{10}}\right\}$

EXERCISE – 2.5

1. Show that vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar if and only if

$\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$, $\vec{c} + \vec{a}$ are coplanar

$\Leftrightarrow [\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}] = 0$

$\Leftrightarrow 2\,[A\ B\ C] = 0$

$\Leftrightarrow \vec{a}, \vec{b}, \vec{c}$ are coplanar

2. The volume of a parallelepiped whose edges are represented by

$-12\vec{i} + m\vec{k}$, $3\vec{j} - \vec{k}$, $2\vec{i} + \vec{j} - 15\vec{k}$ is 546. Find the value of m.

Solution: Let $\vec{a} = -12\vec{i} + m\vec{k}$, $\vec{b} = 3\vec{j} - \vec{k}$, $\vec{c} = 2\vec{i} + \vec{j} - 15\vec{k}$

Volume of the parallelepiped $= [a\ b\ c] = 546$

i.e., $\begin{vmatrix} -12 & 0 & m \\ 0 & 3 & -1 \\ 2 & 1 & -15 \end{vmatrix} = 546$

$-12(-45 + 1) + m\ 90 - 6) = 546$

$= m = -3$

3. Prove that $|[a\ b\ c]| = abc$ if and only if $a, \vec{b}, \vec{c}$ are mutually perpendicular.

Solution: $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular $\Leftrightarrow |[a\ b\ c]|$ is the volume of a cuboids where a, b, c are the co-terminus edges.

$$\Leftrightarrow |[a\ b\ c]| = |\vec{a}|\,|\vec{b}|\,|\vec{c}|$$

$$\Leftrightarrow |[a\ b\ c]| = abc$$

4. Show that the points (1, 3, 1), (1, 1, -1), (-1, 1, 1) (2, 2, -1) are lying on the same plane. (Hint: It is enough to prove any three vectors formed by these four points are coplanar).

Solution: Let $\vec{OA} = \vec{i} + 3\vec{j} + \vec{k}$, $\vec{OB} = \vec{i} + \vec{j} - \vec{k}$, $\vec{OC} = \vec{i} + \vec{j} + \vec{k}$

and $\vec{OD} = 2\vec{i} + 2\vec{j} - \vec{k}$

$$\vec{AB} = \vec{OB} - \vec{OA} = -2\vec{j} - 2\vec{k}$$

$$\vec{AC} = \vec{OC} - \vec{OA} = 2\vec{i} - 2\vec{j}$$

$$\vec{AD} = \vec{OD} - \vec{OA} = \vec{i} - \vec{j} - 2\vec{k}$$

$$[\vec{AB}, \vec{AC}, \vec{AD}] = \begin{matrix} 0 & -2 & -2 \\ -2 & -2 & 0 \\ 1 & -1 & -2 \end{matrix} = 0$$

Hence the above points are lying on the same plane.

5. If $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{b} = -2\vec{i} + 5\vec{k}$, $\vec{c} = \vec{j} - 3\vec{k}$

Verify that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c}$

Solution:

$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & 5 \\ 0 & +1 & -3 \end{vmatrix} = 5\vec{i} - 6\vec{j} - 2\vec{k}$

$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ -5 & -6 & -2 \end{vmatrix}$

$= 12\vec{i} + 9\vec{j} + 3\vec{k}$

$(\vec{a}.\vec{c}) = (2(0) + 3(1) + (-1)(-3)) = 6$

$(\vec{a}.\vec{c})\,\vec{b} = -12\vec{i} + 30\vec{k}$

$(\vec{a}.\vec{b}) = \{(2)(-2) + (3)(0) + (-1)(5)\} = -9$

$(\vec{a}.\vec{b})\,\vec{c} = -9\vec{j} + 27\vec{k}$

$(\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.c)\,c = -12\vec{i} + 9\vec{j} + 3\vec{k}$

Hence $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c}$

6. Prove that $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = o$

Solution:

LHS $= \vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})$

$= (\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c} + (b.\vec{a})\,\vec{c} - (b.c)\,a$

$+ (\vec{c}.\vec{b})\,\vec{a} - (\vec{c}.\vec{a})\,\vec{b}$

$= 0$ R. H. S.

7. If $\vec{a} = 2\vec{i} + 3\vec{j} - 5\vec{k}$, $\vec{b} = -\vec{i} + \vec{j} + 2\vec{k}$ and

$\vec{c} = 4\vec{i} - 2\vec{j} + 3\vec{k}$, show that $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

solution :

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -5 \\ -1 & 1 & 2 \end{vmatrix} = 11\vec{i} + \vec{j} + 5\vec{k}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 11 & 1 & 5 \\ -1 & 1 & 2 \end{matrix} = 13\vec{i} - 13\vec{j} - 26\vec{k}$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ - & 1 & 2 \\ 4 & -2 & 3 \end{vmatrix} = 7\vec{i} + 11\vec{j} - 2\vec{k}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -5 \\ 7 & 11 & -2 \end{vmatrix} = 49\vec{i} - 31\vec{j} + \vec{k}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$$

8. prove that $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$ iff a and c are collinear .

Where the vector triple product is non zero .

Solution : given $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$

$\Leftrightarrow$ $(\vec{a} . \vec{b})\vec{c} - (\vec{b} . \vec{c})\vec{a} = (\vec{a} . \vec{c})\vec{b} - (\vec{a} . \vec{b})\vec{c}$

$\Leftrightarrow$ $(a . b)\, c = (b . c)\, a$

$\Leftrightarrow a = \left(\frac{a \cdot b}{c \cdot b}\right) . c$

$\Leftrightarrow$ a and c are collinear .

9. For any vector $\vec{a}$

Prove that $\vec{i} \times (\vec{a} \times \vec{i}) + \vec{j} \times (\vec{a} \times \vec{j}) + \vec{k} \times (\vec{a} \times \vec{k}) = 2\vec{a}$

Solution:

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{i} \times (\vec{a} \times \vec{b}) = (\vec{i} . \vec{i})\vec{a} - (\vec{i} . \vec{a})\vec{i} = \vec{a} - a_1\vec{i}$

$\vec{j} \times (\vec{a} \times \vec{j}) = (\vec{j} . \vec{j})\vec{a} - (\vec{j} . \vec{a})\vec{j} = \vec{a} - a_2\vec{j}$

$\vec{k} \times (\vec{a} \times \vec{k}) = (\vec{k} . \vec{k})\vec{a} = (\vec{k} . a)k = a - a_3k$

L.H.S. $= 3\vec{a} - (a_1\vec{i} + a_2\vec{j} + a_3\vec{k})$

$= 2\vec{a}$ = R.H.S

10. Prove that $(\vec{a} \times \vec{b}) . (\vec{c} \times \vec{d}) + (\vec{b} \times \vec{c}) . (\vec{a} \times \vec{b}) + (\vec{c} \times \vec{a}) . (\vec{b} \times \vec{d}) = 0$

Solution: $(\vec{a} \times \vec{b}) . (\vec{c} \times \vec{b}) = \begin{vmatrix} \vec{a} & \vec{c} \\ \vec{b} & \vec{c} \end{vmatrix} \begin{vmatrix} \vec{a} & \vec{d} \\ \vec{b} & \vec{d} \end{vmatrix}$

$= (\vec{a} . \vec{c})(\vec{b} . \vec{d}) - (\vec{b} . \vec{c})(\vec{a} . \vec{d})$

$(\vec{b} \times \vec{c}) . (\vec{a} \times \vec{d}) = \begin{vmatrix} \vec{b} & \vec{d} \\ \vec{c} & \vec{a} \end{vmatrix} \quad \begin{vmatrix} \vec{b} & \vec{d} \\ \vec{c} & \vec{d} \end{vmatrix}$

$= (\vec{b} . \vec{a})(\vec{c} . \vec{d}) - (\vec{c} . \vec{a})(\vec{b} . \vec{d})$

$(\vec{c} \times \vec{a}) . (\vec{b} \times \vec{d}) = \begin{vmatrix} \vec{c} & \vec{b} \\ \vec{a} & \vec{b} \end{vmatrix} \quad \begin{vmatrix} \vec{c} & \vec{d} \\ \vec{a} & \vec{d} \end{vmatrix}$

$= (\vec{c}.\vec{b})(\vec{a}.\vec{d}) - (\vec{a}.\vec{b})(\vec{c}.\vec{d})$

L.H.S. = (a. c) (b. d) - (b. c) (a. d)

$+ (\vec{b}.\vec{a})\ (\vec{c}.\vec{d})$ - (c. a) (b. d)

$(\vec{a}.\vec{b})(\vec{a}.\vec{d}) - (\vec{a}.\vec{b})(\vec{c}.\vec{d})$

= 0 = R.H.S

11. Find (a x b). (c x $\vec{d}$) if $\vec{a} = \vec{i} + \vec{j} + \vec{k}$

$\vec{b} = 2\vec{i} + \vec{k},\ \vec{c} = 2\vec{i} + \vec{j} + \vec{k},\ \vec{d} = \vec{i} + \vec{j} + 2\vec{k}$

Solution:

$(\vec{a} \times \vec{b}).\ (\vec{c} \times \vec{d}) = (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{a}.\vec{d})(\vec{b}.\vec{c})$

$\vec{a}.\vec{c} = 2 + 1 + 1 = 4$

$\vec{b}.\vec{d} = 2 + 0 + 2 = 4$

$\vec{a}.\vec{d} = 1 + 1 + 2 = 4$

$\vec{b}.\vec{c} = 4 + 1 = 5$

L.H.S = (4) (4) - (4) (5) = -4

12. Verify $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{b}\ \vec{c}]\ \vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\ \vec{a}$

for $\vec{a}$, $\vec{b}$, $\vec{c}$ and d in problem 11.

Solution:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \vec{i} + \vec{j} - 2\vec{k}$$

$$\vec{c} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \vec{i} - 3j + k$$

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ 1 & -3 & 1 \end{vmatrix} = -5\vec{i} - 3\vec{j} - 4\vec{k}$$

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 1$$

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2$$

$$[\vec{a}\ \vec{b}\ \vec{c}]\ \vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\ \vec{d} = (-4\vec{i} - 2\vec{j} - 2\vec{k}) - (\vec{i} + \vec{j} + 2\vec{k})$$

$$= -5\vec{i} - 3\vec{j} - 4\vec{k}$$

From (1) and (2)

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{b}\ \vec{c}]\ \vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\ \vec{d}$$

<u>EXERCISE – 2.6</u>

1. Find the d.c.s of a vector whose direction rations are 2, 3, - 6.

Solution:

$$\vec{r} = \sqrt{(2)^2 + (3)^2 + (-6)^2} = \sqrt{49} = 7$$

d.c.s are $\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}$

2. (i) Can a vector have direction angles 30°, 45°, 60°.

(ii) Can a vector have direction angles 45°, 60°, 120°?

Solution:

(i) For direction angles $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\text{Cos}^2 30 + \cos^2 45 + \cos^2 60$

$= \frac{3}{4}\frac{1}{2}\frac{1}{4} \neq 1$

$\therefore$ 30°, 45°, 60° are not possible to be direction angles.

(ii) $\cos^2 45 + \cos^2 60 + \cos^2 120 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$, $\therefore$ yes

3. What are the d.c.s of the vector equally inclined to the axes?

Solution:

$\text{Cos}^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ But $\alpha\beta = \gamma$

$\therefore \cos^2\gamma = \frac{1}{3}$ => $\cos \alpha \frac{1}{\sqrt{3}}$

$\therefore$ The d.c. 's are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

4. A vector $\vec{r}$ has length $35\sqrt{2}$ and direction ratios (3, 4, 5) find the direction cosines and components of $\vec{r}$.

Solution:

The direction rations are (3, 4, 5)

$\sqrt{3^2+4^2+5^2} = \sqrt{50} = 5\sqrt{2}$

d.c.'s are $\left(\frac{3}{5\sqrt{2}}\frac{4}{5\sqrt{2}}\frac{5}{5\sqrt{2}}\right)$

$\vec{r} = 35\sqrt{2}\left(\frac{3\vec{i}+4\vec{j}+5\vec{k}}{5\sqrt{2}}\right)$

$\vec{r} = 7\ [3\vec{i} + 4\vec{j} + 5\vec{k}] = 21\vec{i} + 28\vec{j} + 35\vec{k}$

5. Find direction cosines of the line joining (2, -3, 1) and (3, 1, -2).

Solution:

$\vec{r} = \vec{a} + 1\ (\vec{b} - \vec{a})$

$\vec{r} = 2\vec{i} - 3\vec{j} + \vec{k} + 1\ (-\vec{i} - 4\vec{j} + 3\vec{k})$

$\therefore$ d.r.'s are (-1, -4, 3) => $\vec{r} = \sqrt{(-1)^2+(-4)^2+3^2} = \sqrt{26}$

Direction cosines $\pm\left(\frac{-1}{\sqrt{26}}, \quad \frac{-4}{\sqrt{26}}, \quad \frac{3}{\sqrt{26}}\right)$

Note: Since any one point can take as the first point, we have directions cosines are $\pm$ ( )

6. Find the vector and Cartesian equation of the line through the point (3, -4, -2) and parallel to the vector $9\vec{i} + 6\vec{j} + 2\vec{k}$.

Solution:

Vector equation:

$\vec{r} = \vec{a} + i\vec{b}$ where $\vec{a} = 3\vec{i} - 4\vec{j} - 2\vec{k}$, $\vec{b} = 9\vec{i} + 6\vec{j} + 2\vec{k}$

$\vec{r} = (3\vec{i} - 4\vec{j} - 2\vec{k}) + t\ (9\vec{i} + 6\vec{j} + 2\vec{k})$

Cartesian form:

$$\frac{x-x_1}{l} = \frac{y-y}{m} = \frac{z-z_1}{n}$$

Where $(x_1, y_1, z_1) = (3, -4, -2)$

$(l, m, n) = (9, 6, 2)$

The equation of the line is

$$\frac{x-3}{9} = \frac{y+4}{6} = \frac{z+2}{2}$$

7. Find the vector and Cartesian equation of the line joining the points (1, -2, 1) and (0, -2, 3)

Solution:

Vector equation: $\vec{r} = \vec{a} + t\,(\vec{b} - \vec{a})$

Where $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$

$\vec{b} = 2\vec{j} + 3\vec{k}$

$\vec{b} - \vec{a} = \vec{i} + 2\vec{k}$

$\vec{r} = (\vec{i} - 2\vec{j} + \vec{k}) + r\,(-\vec{i} + 2\vec{k})$

(or) $\vec{r} = (1 - t)\,\vec{a} + t\vec{b}$

i.e., $\vec{r} = (1 - t)\,(\vec{i} - 2\vec{j} + \vec{k}) + t\,(-2\vec{j} + 3\vec{k})$

Cartesian form:

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2 y} = \frac{z-z}{z_2-z_1}$$

Here $(x_1, y_1, z_1) = (1, -2, 1)$; $(x_2, y_2, z_2) = (0, -2, 3)$

The equations is $\frac{x-1}{-1} = \frac{y+2}{0} = \frac{z-1}{2}$

8. Find the angle between the following lines.

$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-4}{6}$ and $x + 1 = \frac{y+2}{2} = \frac{z-4}{2}$

Solution:

The parallel vectors to the lines are $\vec{u} = 2\vec{i} + 3\vec{j} + 6\vec{k}$ and $\vec{v} = \vec{i} + 2\vec{j} + 2\vec{k}$ respectively

Let θ be the angle between the given lines

$$\cos\theta = \frac{\vec{u}\,\vec{v}}{|\vec{u}|\,|\vec{v}|}$$

$\vec{u}.\vec{v} = 20$; $|\vec{u}| = 7$, $|\vec{v}| = 3$

$$\cos\theta = \left(\frac{20}{21}\right)$$

$$\theta = \cos^{-1}\frac{20}{21}$$

9. Find the angle between the lines

$\vec{r} = 5\vec{i} - 7\vec{j} + \mu(-\vec{i} + 4\vec{j} + 2\vec{k})$

$\vec{r} = 2\vec{i} + \vec{k} + \mu(3\vec{i} + 4\vec{k})$

Solution:

The parallel vectors to the lines are

$\vec{u} = -\vec{i} + 4\vec{j} + 2\vec{k}$ and $\vec{v} = 3\vec{i} + 4\vec{k}$ respectively

Let θ be the angle between the given lines.

$\cos\theta \frac{\vec{u}\ \vec{v}}{|\vec{u}|\ |\vec{v}|}$

$\vec{u}.\vec{v} = 5;\quad |\vec{u}| = \sqrt{21}.\quad |\vec{v}| = 5$

$\cos\theta = \frac{5}{\sqrt{215}} = \frac{1}{\sqrt{21}}$

$\theta = \cos^{-1}\frac{1}{\sqrt{21}}$

EXERCISE – 2.7

1. Find the shortest distance between the parallel lines

(i) $\vec{r} = (2\vec{i} + \vec{j} - \vec{k}) + t(\vec{i} - 2\vec{j} + 3\vec{k})$

$\vec{r} = (\vec{i} - 2\vec{j} + \vec{k}) + s(\vec{i} - 2\vec{j} + 3\vec{k})$

(ii) $\frac{x-1}{-1} = \frac{y}{3} = \frac{z+3}{2}$ and $\frac{x-3}{-1} = \frac{y+1}{3} = \frac{z-1}{2}$

Solution:

(i) Let $\vec{u} = \vec{i} - 2\vec{j} + 3\vec{k}$. $\vec{a_1} = 2\vec{i} - \vec{j} - \vec{k}$ and $\vec{a_2} = \vec{i} - 2\vec{j} + \vec{k}$

Shortest distance between the lines $d = \frac{u \times (a_2 - a_1)}{|u|}$

$\vec{u} \times (\vec{a_2} - \vec{a_1}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ -1 & -1 & 2 \end{vmatrix} = -\vec{i} - 5\vec{j} - 3\vec{k}$

$|\vec{u} \times (\vec{a_2} - \vec{a_1})| = \sqrt{(-1)^2(-5)^2 + (-3)^2} = \sqrt{35}$

$|\vec{u}| = \sqrt{14}$

$\therefore d = \frac{\sqrt{35}}{\sqrt{14}} = \sqrt{\frac{5}{2}}$

(ii) Let $\vec{u} = -\vec{i} + 3\vec{j} + 2\vec{k}$ and $\vec{a_1} = \vec{i} - 3\vec{k}$

$|\vec{u}| = \sqrt{14}$ $\qquad \vec{a_2} = 3\vec{i} - \vec{j} + \vec{k}$

$\vec{a_2} - \vec{a_1} = 2\vec{i} - \vec{j} - 4\vec{k}$

$$\vec{u} \times (\vec{a_2} - \vec{a_1}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 3 & 2 \\ 2 & -1 & -4 \end{vmatrix} = 14\vec{i} + 8\vec{j} - 5\vec{k}$$

$|\vec{u} \times (\vec{a_2} - \vec{a_1})| = \sqrt{285}$

$\therefore d = \frac{\sqrt{285}}{\sqrt{14}} = \sqrt{\frac{285}{14}}$

2. Show that the following two lines are skew lines:

$\vec{r} = (3\vec{i} + 5\vec{j} + 7\vec{k}) + t(\vec{i} - 2j + k)$ and

$\vec{r} = (\vec{i} + \vec{j} + \vec{k}) + s(\vec{7i} + \vec{6j} + \vec{7k})$

Solution: Compare the given lines with

$\vec{r} = \vec{a_1} + t\vec{u}$ and $\vec{r} = \vec{a_2} + s\vec{v}$

$\vec{u} = \vec{i} - 2\vec{j} + \vec{k}$ $\qquad \vec{a_1} = 3\vec{i} + 5\vec{j} + 7\vec{k}$

$\vec{v} = 7i + 6j + 7k$ $\qquad \vec{a_2} = \vec{i} + \vec{j} + \vec{k}$

$\vec{a}_2 - \vec{a}_1 = -2\vec{i} - 4\vec{j} - 6\vec{k}$

$$[(\vec{a}_2 - \vec{a}_1)\ \vec{u}\ \vec{v}] = \begin{vmatrix} -2 & -4 & -6 \\ 1 & -2 & 1 \\ 7 & 6 & 7 \end{vmatrix} = 2(-20) + 4(0) - 6(20)$$

$= -80 \neq 0$ $\therefore$ The above lines are skew lines.

3. Show that the lines $\frac{x-1}{1} = \frac{y+1}{-1} = \frac{z}{3}$ and $\frac{x-2}{1} = \frac{y-1}{2} = \frac{-z-1}{1}$ intersect and ind their point of intersection.

Solution: Condition for intersecting is d = 0

(i.e.,0 $[(a_2 - a_1)\ u\ v)] = 0$ or $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$

Here $(x_1, y_1, z_1) = (1, -1, 0)$

$(x_2, y_2, z_2) = (2, 1, -1)$

$(l_1, m_1, n_1) = (1, -1, 3)$

$(l_2, m_2, n_2) = (1, 2, -1)$

$$[(\vec{a}_2 - \vec{a}_1)\ \vec{u}\ \vec{v}] = \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 3 \\ 1 & 2 & -1 \end{vmatrix} = 5 + 8 - 3 = 0$$

Further $\vec{u}$ and $\vec{v}$ are not parallel.

$\therefore$ The lines intersect For point of intersection, take $\frac{x-1}{1} = \frac{y+1}{-1} = \frac{z}{3}$ =

Any point on this line is of the form (m + 1, -m - 1. 3m). $\frac{x-2}{1}$ = $\frac{y-1}{2}$ = $\frac{z+1}{-1}$ = μ.

Any point on this line is of the form (μ + 2, 2μ + 1, -μ – 1)

(m + 1, - m - 1, 3m) = (μ + 2, 2μ + 1, -μ - 1)

$m + 1 = \mu + 2$

m - μ = 1

$-m$ - 1 = 2 μ = 2

m - 2 μ = 2

Solving (1) and (2), μ = -1, m = 0

∴ To get the point of intersection either put μ = - 1 or m = 0

∴ The point of intersection is (1, -1, 0)

4. Find the shortest distance between the skew lines

$\frac{x-6}{3}$ = $\frac{y-7}{-1}$ = $\frac{z-4}{1}$

and $\frac{x}{-3}$ = $\frac{y+9}{2}$ = $\frac{z-2}{4}$

Solution:

Shortest distance $\vec{d}$ = $\frac{|[\vec{a_2}-\vec{a_1})\ \vec{u}\ \vec{v}}{|\vec{u}\ x\ \vec{v}|}$

$\vec{u}$ = 3$\vec{i}$ - $\vec{j}$ + $\vec{k}$ $\vec{a_1}$ = 6$\vec{j}$ + 7$\vec{j}$ + 4$\vec{k}$

$\vec{v}$ = 3$\vec{i}$ + 2$\vec{j}$ + 4$\vec{k}$ $\vec{a_2}$ = 9$\vec{j}$ + 2$\vec{k}$

$$\vec{a_2} - \vec{a_1} = -6\vec{i} - 16\vec{j} - 2\vec{k}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix} = -6\vec{i} - 15\vec{j} + 3\vec{k}$$

$$|\vec{u} \times \vec{v}| = \sqrt{270}$$

$$[(\vec{a_2} - \vec{a_1})\ \vec{u}\ \ \vec{v}] = \begin{vmatrix} -6 & -16 & -2 \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix}$$

or $(\vec{a_1} - \vec{a_1})\ (\vec{u} \times \vec{v}) = 36 + 240 - 6 = 270$

$\therefore\quad d = \frac{\vec{270}}{\sqrt{270}} = \sqrt{270}$

$= 3\sqrt{30}$

5. Show that (2, -1, 3), (1, -1, 0) and (3, -1, 6) are collinear.

Solution:

The equation passing through (2, -1, 3) and (1, -1, 0) is

$$\frac{x-2}{-1} = \frac{y+1}{0} = \frac{z-3}{-3}\quad m \text{ (say)}$$

Any point on this line is of the form (-m + 2, -1, -3m + 3)

The point (3, -1, 60 is obtained by putting m = -1)

$\therefore$ The third point lies on the same line. Hence three points are collinear.

6. If the points (m, 0, 3), (1, 3, -1) and 9-5, -3, 7) are collinear then find m.

Solution:

Since the three points are collinear, the position vector of three points are coplanar.

Let $\vec{a} = m\vec{i} + 3\vec{k}$, $\vec{b} = \vec{i} + 3\vec{j} - \vec{k}$ and $\vec{c} = -5\vec{i} - 3\vec{j} + 7\vec{k}$

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} m & 0 & 3 \\ 1 & 3 & -1 \\ -5 & -3 & 7 \end{vmatrix} = 0$$

$18m + 36 = 0 \Rightarrow m = -2.$

EXERCISE – 2.8

1. Find the vector and Cartesian equations of a plane which is at a distance of 18 units from the origin and which is normal to the vector $2\vec{i} + 7\vec{j} + 8\vec{k}$

Solution: Here $p = 18$ and $\vec{n} = 2\vec{i} + 7\vec{j} + 3\vec{k}$

$$\therefore\ n = \frac{n}{|n|} = \frac{2\vec{i} + 7\vec{j} + 8\vec{k}}{\sqrt{117}}$$

Hence the required vector equation of the plane is $\vec{r}.\vec{n} = p$

$$\vec{r}\ \frac{2\vec{i} + 7\vec{j} + 8\vec{k}}{\sqrt{117}} = 18$$

Cartesian form:

$\vec{r}.\ (2\vec{i} + 7\vec{j} + 9\vec{k}) = 18\sqrt{117}$

$r.\ (2i + 7j + 8k) = 54\sqrt{13}$

$(x\vec{i} + y\vec{j} + z\vec{k}).\ (2\vec{i} + 7\vec{j} + 8\vec{k}) = 54\sqrt{13}$ i.e., $2x + 7y + 8z = 54\sqrt{13}$

2. Find the unit normal vectors to the plane 2x – y + 2z =5.

Solution:

$2x - y + 2z = 5 \Leftrightarrow (x\vec{i} + y\vec{j} + z\vec{k}).\ (2\vec{i} - \vec{j} + 2\vec{k}) = 5$

Here $\vec{n} = 2\vec{i} - \vec{j} + 2\vec{k}$

Unit normal vectors $\pm\ n = \pm\frac{\vec{n}}{|\vec{n}|} = \pm\frac{2\vec{i} - \vec{j} + 2\vec{k}}{3}$

3. Find the length of the perpendicular from the origin to the plane

$\vec{r}.\ (3\vec{i} + 4\vec{j} + 12\vec{k}) = 26$

Solution: Write the given equation in the form of $\vec{r}.\ \vec{n} = p$

Given $\vec{r}.\ (3\vec{i} + 4\vec{j} + 12\vec{k}) = 26 \Rightarrow \vec{r}.\left(\frac{3\vec{i} + 4\vec{j} + 12\vec{k}}{\sqrt{169}}\right) = \frac{26}{\sqrt{169}}$

$\Rightarrow \vec{r}\ .\ \left(\frac{3\vec{i} + 4\vec{j} + 12\vec{k}}{13}\right) = 2$

∴ Length of the perpendicular from origin p = 2

4. The foot of the perpendicular draw from the origin to a plane is

(8, - 4, 3) . Find the equation of the plane.

Solution:

The required plane passing through the point a (8, -4, 3) and is perpendicular

to OA

$\therefore \vec{a} = 8\vec{i} - 4\vec{j} + 3\vec{k}$ and $\vec{n} = \vec{OA} = 8\vec{i} - 4\vec{j} + 3\vec{k}$

$\therefore$ the required equation of the plane is $\vec{r}.\vec{n} = \vec{a}.\vec{n}$

$\vec{r}.(8\vec{i} - 4\vec{j} + 3\vec{k}) = (8\vec{i} - 4\vec{j} + 3\vec{k}).(8\vec{i} - 4\vec{j} + 3\vec{k})$

The vector form is $\vec{r}.(8\vec{i} - 4\vec{j} + 3\vec{k}) = 89$

Cartesian form: $(x\vec{i} + y\vec{j} + z\vec{k}).(8\vec{i} - 4\vec{j} + 3\vec{k}) = 89$

=> $8x - 4y + 3z = 89$

5. Find the equation of the plane through the point whose p.v. is

$2\vec{i} - \vec{j} + \vec{k}$ and perpendicular to the vector $4\vec{i} + 2\vec{j} - 3\vec{k}$.

Solution:

The required equation of the plane through $2\vec{i} - \vec{j} + \vec{k}$ and perpendicular to $4\vec{i} + 2\vec{j} - 3\vec{k}$ is

$\vec{r}.\vec{n} = \vec{a}.\vec{n}$

Here $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$ and $\vec{n} = 4\vec{i} + 2\vec{j} - 3\vec{k}$

$\vec{r}\cdot(4\vec{i}+2\vec{j}-3\vec{k}) = (2\vec{i}-\vec{j}+\vec{k})\ \ (4\vec{i}+2\vec{j}-3\vec{k})$

i.e., $r\,(4\vec{i}+2\vec{j}-3\vec{k}) = 3$

The Cartesian form is $(x\vec{i} + y\vec{j} + 2\vec{k})\ \ (4\vec{i} + 12\vec{j} - 3\vec{k}) = 3$

EXERCISE – 2.9

1. Find the equation of the plane which contains the two lines

$\frac{x+1}{2} = \frac{y-2}{-3} = \frac{z-3}{4}$ and $\frac{x-4}{3} = \frac{y-1}{2} = z-8$

Solution:

The required equation of the plane through A (-1, 2, 3) and parallel to

$\vec{u} = 2\vec{i} - 3\vec{j} + 4\vec{k}$ and $\vec{v} = 3\vec{i} + 2\vec{j} + 1\vec{k}$

The required equation is $\vec{r} = \vec{a} + s\vec{u} + t\vec{v}$

$\vec{r} = (-\vec{i} + 2\vec{j} + 3\vec{k}) + s(2\vec{i} - 3\vec{j} + 4\vec{k}) + t(3\vec{i} + 2\vec{j} + \vec{k})$

Cartesian form:

(x_1, y_1, z_1) is (-1, 2, 3); (l_1, m_1, n_1) is (2, -3, 4) (l_2, m_2, n_2) is (3, 2, 1)

The equation of the plane is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

i.e., $\begin{vmatrix} x+1 & y-2 & z-3 \\ 2 & -3 & 4 \\ 3 & 2 & 1 \end{vmatrix} = 0$

=> 11x - 10y – 13z + 70 = 0

This is the required equation in Cartesian form.

Note: The above plane can be determined by passing through

(-1, 2, 3), (4, 1, 8) and parallel to $2\vec{i} - 3\vec{j} + 4\vec{k}$ or $3\vec{i} + 2\vec{j} + \vec{k}$

2. Can you draw a plane through the given two lines? Justify your answer.

$\vec{r} = (\vec{i} + 2\vec{j} - 4\vec{k}) + t(2\vec{i} + 3\vec{j} + 6\vec{k})$ and

$\vec{r} = (3\vec{i} + 3\vec{j} + 5\vec{k}) + s(2\vec{i} + 3\vec{j} + 8\vec{k})$

Solution:

Comparing with $\vec{r} = \vec{a_1} + t\vec{u}$; $\vec{r} = \vec{a_2} + s\vec{v}$ we get

$\vec{a_1} = \vec{i} + 2\vec{j} - 4\vec{k}$

and $\vec{a_2} = 3\vec{i} + 3\vec{j} - 5\vec{k}$

$\vec{u} = 2\vec{i} + 3\vec{j} + 6\vec{k}$

and $\vec{v} = -2\vec{i} + 3\vec{j} + 8\vec{k}$

$[(\vec{a_2} - \vec{a_1})\ \vec{u}\ \vec{v}] = \begin{vmatrix} 2 & 1 & -1 \\ 2 & 3 & 6 \\ -2 & 3 & 8 \end{vmatrix} = -28 \neq 0$

These lines are not intersecting and u, v are not parallel.

∴ they are skew lines. We can't draw a plane through the given two lines.

3. Find the point of intersection of the line

$\vec{r} = (\vec{j} - \vec{k}) + s(2\vec{i} - \vec{j} + \vec{k})$ and xz - p-lane

Solution:

Cartesian equation of the given line is $\frac{X-0}{2} = \frac{Y-1}{-1} = \frac{Z+1}{1}$

Equation of xz plane is y = 0

$\therefore \frac{x}{2} = \frac{-1}{-1} = \frac{z+1}{1}$ => x = 2, z = 0

∴ The required point is (2, 0, 0)

4. Find the meeting point of the line

$\vec{r} = (2\vec{i} + \vec{j} - 3\vec{k}) + t(2\vec{i} - \vec{j} - \vec{k})$ and the plane

x – 2y + 3z + 7 = 0

Solution:

Cartesian form of the line is $\frac{x-2}{2} = \frac{y-2}{-1} = \frac{z+3}{-1}$ = m (say)

Any point on this line is of the form (2m + 2, -m, -m - 3)

This point lie on the plane x – 2y + 3y + 7 = 0

(2m + 20 -2 (-m + 1) + 3 (-m - 3) + 7 = 0

=> m = 2

∴ The point is (6, -4, -5)

5. Find the distance from the origin to the plane

$\vec{r}.\ (2\vec{i} - \vec{j} + 5\vec{k}) = 7$

Solution:

Cartesian form of the plane is 2x – y + 5z – 7 = 0

Distance from the origin to the plane ax + by + cz + d = 0 is

$| \frac{d}{\sqrt{a^2+b^2+c^2}}$

$= \frac{-7}{\sqrt{30}} = \frac{7}{\sqrt{30}}$

6. Find the distance between the parallel planes

Solution:

Distance between two parallel planes

$ax + by + cz + d_1 = 0$

$ax + by + cz + d_2 = 0$

$\vec{d} = \frac{|\vec{d_1} - \vec{d_2}|}{\sqrt{a^2+b^2+c^2}}$

The given planes are $x - y + 3z + 5 = 0$ and $x - y + 3z + \frac{7}{2} = 0$

$$\vec{d} = \frac{|5-\frac{7}{2}}{\sqrt{(1)^2+(-1)^2+(3)^2}} = \frac{\frac{3}{2}}{\sqrt{11}} = \frac{3}{2\sqrt{11}}$$

EXERCISE – 2.10

1. Find the angle between the following planes:

(i) $2x + y - z = 9$ and $x + 2y + z = 7$

(ii) $2x - 3y + 4z = 1$ and $-x + y = 4$

(iii) $\vec{r} \cdot (3\vec{i} + \vec{j} + \vec{k}) = 7$ and $\vec{r} \cdot (\vec{i} + 4j - 2\vec{k}) = 10$

Solution:

(i) The normals to the given planes are $n_1 = 2\vec{i} + \vec{j} - \vec{k}$

and $n_2 = \vec{i} + 2\vec{j} + \vec{k}$

Let θ be the angle between the planes then

$$\cos\theta = \frac{\vec{n}_1 \; \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{(2\vec{i}+\vec{j}-\vec{k}).}{\sqrt{6}} \frac{(\vec{i}+\vec{j}-\vec{k})}{\sqrt{6}}$$

$$= \frac{6}{\sqrt{6}\sqrt{6}} = \frac{1}{2}$$

$$=> \theta \frac{\pi}{3}$$

(ii) The normals to the given planes are $n_1 = 2\vec{i} - 3\vec{j} + \vec{k}$

and $n_2 = i + j$

Let θ be the angle between the planes, then

$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|} = \frac{(2\vec{i}+3\vec{j}+\vec{k})\cdot(-\vec{i}+\vec{j})}{\sqrt{29}\;\sqrt{2}}$

$= \frac{-5}{\sqrt{58}} \Rightarrow \theta = \cos^{-1}\frac{-5}{\sqrt{58}}$

(iii) The normals to the given planes are $\vec{n_1} = 3\vec{i} + \vec{j} - \vec{k}$ and

$n_2 = \vec{i} + 4\vec{j} - 2\vec{k}$

Let θ be the angle between the planes then

$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|} = \frac{9}{\sqrt{11}\;\sqrt{21}} = \frac{9}{\sqrt{231}}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{9}{\sqrt{231}}\right)$

2. Show that the following planes are at right angles.

$\vec{r}\cdot(2\vec{i} - \vec{j} + \vec{k}) = 15$ and $\vec{r}\cdot(\vec{i} - \vec{j} - 3\vec{k}) = 3.$

Solution:

The normals to the given plane are

$\vec{n_1} = 2\vec{i} - \vec{j} + \vec{k}$ and $\vec{n_2} = \vec{i} - \vec{j} - 3\vec{k}$

=> The normals are perpendicular.

=> The planes are at right angles.

3. The planes $\vec{r}\cdot(2\vec{i} + \mu\vec{j} - 3\vec{k}) = 10$ and $\vec{r}\cdot(\mu\vec{i} + 3\vec{j} + \vec{k}) = 5$ are perpendicular. Find μ .

Solution:

The normals to the given planes are

$\vec{n_1} = 2\vec{i} + \vec{\mu} j - \vec{3} k$ and $\vec{n_2} = \vec{\mu} i + \vec{3} j + \vec{k}$

Since the planes are perpendicular $\vec{n_1} . \vec{n_2} = 0$

=> $\vec{n_1} . \vec{n_2} = 2\mu + 3\mu - 3 = 0$

=> $5\mu = 9$ => $\mu = \frac{3}{5}$

4. Find the angle between the line $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{-2}$ and the plane

$3x + 4y + z + 5 = 0$

Solution:

The normal to the given plane is $\vec{n} = 3\vec{i} + \vec{4}j + \vec{k}$

The parallel vector to the line $\vec{b} = 3\vec{i} - \vec{j} - 2\vec{k}$

Let θ be the angle between the line and plane. Then

$$\sin\theta = \frac{\vec{b}.\vec{n}}{|\vec{b}|\,|\vec{n}|}$$

$\vec{b} . \vec{n} = (3)(3) + (-1)(4) + (-2)(1)$

$|\vec{b}| = 2\ |\vec{n}| = \sqrt{91}$

$\sin\theta = \frac{3}{2\sqrt{91}}$ => $\theta\ \sin^{-1}\left(\frac{3}{2\sqrt{91}}\right)$

5. Find the angle between the line $\vec{r} = \vec{i} + \vec{j} + 3\vec{k} + \mu\ (2\vec{i} + \vec{j} - \vec{k})$ and the plane $\vec{r}.(\vec{i} + \vec{j}) =$.

Solution:

The normal to the given plane to $\vec{n} = \vec{i} + \vec{j}$ and the parallel vector the line $\vec{b} = 2\vec{i} + \vec{j} - \vec{k}$.

Let θ be the angle between the line and the plane

$\sin \theta \frac{\vec{b}.\ \vec{n}}{|\vec{b}|\ |\vec{n}|}$

$\vec{b}.\ \vec{n} = 3,\ ;\ |\vec{b}| = \sqrt{6}\ ;\ |\vec{n}| = \sqrt{2}$

$\sin \theta \frac{6}{\sqrt{6}\sqrt{2}} = \frac{\sqrt{3}}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$

EXERCISE – 2.11

1. Find the vector equation of a sphere with centre having position vector

$2\vec{i} - \vec{j} + 3\vec{k}$ and radius 4 units. Also find the equation in Cartesian form.

Solution:

Vector equation of a sphere $|\vec{r} - \vec{c}| = a$

Here $c = 2\vec{i} - \vec{j} + 3\vec{k}$ and $a = 4$

$\therefore$ Vector equation is $|\vec{r} - (2\vec{i} - \vec{j} + 3\vec{k})| = 4$

Cartesian form:

Let $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

$\vec{r} - \vec{c} = (x-2)\,\vec{i} + (y+1)\,\vec{j} + (z-3)\,\vec{k}$

$|\vec{r} - \vec{c}|^2 = 4^2 \Rightarrow (x-2)^2 + (y+1)^2 + (z-3)^2 = 16$

$\Rightarrow x^2 + y^2 + z^2 - 4x + 2y - 6z - 2 = 0$

2. Find the vector and Cartesian equation of the sphere on the join of the points A and B having position vectors $2\vec{i} + 6\vec{j} - 7\vec{k}$ and $-2\vec{i} + 4\vec{j} - 3\vec{k}$ respectively as a diameter. Find also the centre and radius of the sphere.

Solution:

Vector equation of a sphere joining the points A and B whose p.v.s. and a and b is $(\vec{r} - \vec{a}).(\vec{r} - \vec{b}) = 0$

Here $\vec{a} = 2\vec{i} + 6\vec{j} - 7\vec{k}$ and $\vec{b} = 2\vec{i} + 4\vec{j} - 3\vec{k}$

$[\vec{r} - (2\vec{i} + 6\vec{j} - 7\vec{k}).]\ [\vec{r} - (-2\vec{i} + 4\vec{j} - 3\vec{k})] = 0$

Cartesian form:

Let $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

$\vec{r} - \vec{a} = (x-2)\,\vec{i} + (y-6)\,\vec{j} + (z+7)\,\vec{k}$

$\vec{r} - \vec{b} = (x+2)\,\vec{i} + (y-4)\,\vec{j} + (z+3)\,\vec{k}$

$(\vec{r} - \vec{a}).(\vec{r} - \vec{b}) = 0$

$\Rightarrow (x-2)(x+2) + (y-6)(y-4) + (z+7)(z+3) = 0$

$\Rightarrow x^2 + y^2 + z^2 - 10y + 10z + 41 = 0$

Compare with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

u = 0. v = - 5, w = 5, d = 41

Centre is (-u, -v, -w) = (0, 5, -5)

radius is $= \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{25 + 25 - 41} = 3$

3. Obtain the vector and Cartesian equation of the sphere whose centre is 91, -1, 1) and radius is the same as that of the sphere

$|\vec{r} - (\vec{i} + \vec{j} + 2\vec{k})| = 5.$

Solution:

Vector equation of sphere $|\vec{r} - \vec{c}| = a$

Here $\vec{c} = \vec{i} - \vec{j} + \vec{k}$, a = 5

$\therefore$ Vector equation is $| r - (\vec{i} - \vec{j} + \vec{k})| \vec{=} 5$

Cartesian form:

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and centre (1, - 1, 1), a = 5

$(x-1)^2 + (y+1)^2 + (z-1)^2 = 5^2$

$\Rightarrow x^2 + y^2 + z^2 - 2x + 2y - 2z - 22 = 0$

4. If A (-1, 4, -3) is one end of a diameter AB of the sphere

$x^2 + y^2 + z^2 - 3x - 2y + 2z - 15 = 0$, the find the coordinates of B.

Solution:

Comparing with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$u = -\frac{3}{2}$, $v = -1$, $w = 1$

Centre of the sphere is $\left(\frac{-3}{2},\ 1\ -1\right)$

One end of the diameter is (-1, 4, -3)

Let B (x_2, y_2, z_2) be the other end of the diameter.

The mid point of AB is th centre $\left(\frac{-3}{2},\ 1\ -1\right)$

i.e., $\left(\frac{-1+x_2}{2}, \frac{4+y_2}{2}, \frac{-3+z_2}{2}\right) = \left(\frac{-3}{2},\ 1\ -1\right)$

=> $x_2 = 4$, $y_2 = -2$, $z_2 = 1$

∴ The co-ordinates of B are (4, -2, 1)

5. Find the centre and radius of each of the following spheres.

(i) $|\vec{r} - (2\vec{i} - \vec{j} + 4\vec{k}| = 5$

(ii) $|2\vec{r} + (3\vec{i} - \vec{j} + 4\vec{k})| = 4$

(iii) $x^2 + y^2 + z^2 + 4x - 8y + 2z = 5$

(iv) $r^2 - \vec{r}.(4\vec{i} + 2\vec{j} - 6\vec{k}) - 11 = 0$

Solution:

(i) Vector equation of sphere is $|\vec{r} - (2\vec{i} - \vec{j} + 4\vec{k})| = 5$

∴ Centre is (2, -1, 4) and radius is 5.

(ii) Vector equation of sphere $|2\vec{r} + (3\vec{i} - \vec{j} + 4\vec{k})| = 4$

$\Rightarrow |2\vec{r} - (3\vec{i} + \vec{j} - 4\vec{k})| = 4$

$\Rightarrow |\vec{r} - \frac{1}{2}(-3\vec{i} + \vec{j} - 4\vec{k})| = 2$

$\Rightarrow$ Centre is $\left(\frac{-3}{2}, \frac{1}{2} - 2\right)$ and radius is 2

(iii) Cartesian equation of sphere $x^2 + y^2 + z^2 + 4x - 8y + 2z = 5$

$u = 2, v = -4, w = 1, d = -5$

centre $(-u, -v, -w) = (-2, 4, -1)$

radius $= \sqrt{u^2+v^2+w^2} - d = \sqrt{4 + 16 + 1 + 5} = \sqrt{26}$

(iv) Equation of sphere $r^2 - \vec{r}.(4\vec{i} + 2\vec{j} - 6\vec{k}) - 11 = 0$

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$(x\vec{i} + y\vec{j} + z\vec{k})^2 - (x\vec{i} + y\vec{j} + z\vec{k}).(4\vec{i} + 2\vec{j} - 6\vec{k}) - 11 = 0$

$\Rightarrow x^2 + y^2 = z^2 - (4x + 2y - 6z) - 11 = 0$

$\Rightarrow x^2 + y^2 = z^2 - 4x - 2y + 6z + 11 = 0$

Here $u = -2, v = --1, w = 3, d = -11$

Centre is $(-u, -v, -w) = (2, 1, -3)$

Radius $= \sqrt{u^2 + v^2 + w^2} - d = 5$

6. Show that diameter of a sphere subtends a right angle at a point on the surface.

Solution:

Let P be a point on the surface of the sphere and AB be a diameter. Consider the great circle on the sphere passing through the points P, A and B. Take the centre O as the point of reference.

$\vec{PB} = \vec{OB} - \vec{OP}$

$\vec{AP} = \vec{OP} - \vec{OA} = \vec{OP} + \vec{OB}$

$\vec{AP} \cdot \vec{PB} = (\vec{OP} + \vec{OB}) \cdot (\vec{OB} - \vec{OP}) = |OP|^2 - |OB|^2$

$= 0$ SINCE $|\vec{OP}| = |\vec{OB}|$

∴ AB subtends a right angle at P o the surface.

Hence the result.